NAME (print): ________________________

Topology Ph.D. Entrance Exam, August 2011

Write a solution of each exercise on a separate page.

Solve EACH of the exercises 1-3

Ex. 1. Let X and Y be Hausdorff topological spaces and let $f\colon X \to Y$ be continuous. Answer YES or NO for each of the following questions. In case your answer is "NO" give a counterexample for the statement. In case your answer is "YES" give a short argument. (Answer: "standard theorem" is acceptable, when appropriate.)

(a) If A is a compact subset of X, then $f[A]$ is a compact subset of Y.

(b) If A is a closed subset of X, then $f[A]$ is a closed subset of Y.

(c) If B is a compact subset of Y, then $f^{-1}(B)$ is a compact subset of X.

(d) If B is a closed subset of Y, then $f^{-1}(B)$ is a closed subset of X.

Ex. 2. Let $\langle X, \mathcal{T}_1\rangle$ and $\langle Y, \mathcal{T}_2\rangle$ be topological spaces.

(a) Define the product topology on $Z = X \times Y$.

(b) Prove that $\mathrm{cl}(A) \times \mathrm{cl}(B) = \mathrm{cl}(A \times B)$ for every $A \subset X$ and $B \subset Y$.

Ex. 3. Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces. Prove that the following two definitions of continuity of $f\colon X \to Y$ are equivalent:

(a) (topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.

(b) (ε-δ definition) For every $x_0 \in X$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x \in X$, if $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \varepsilon$.

Solve TWO of the exercises 4-6

Ex. 4. Let X and Y be Hausdorff topological. Recall that a graph of a function $f: X \to Y$ is defined as $G(f) = \{\langle x, f(x) \rangle \in X \times Y : x \in X\}$ and that, for a metric space $\langle Z, d \rangle$ and non-empty sets $A, B \subset Z$, their distance is defined as $\text{dist}(A, B) = \inf\{d(a, b) : a \in A \ \& \ b \in B\}$.

(a) Show that if f is continuous, then its graph $G(f)$ is a closed subset of $X \times Y$.

(b) Show that if $f, g: [0, 1] \to [0, 1]$ are continuous functions, then

$$\text{dist}(G(f), G(g)) = 0 \text{ if, and only if, } f(x) = g(x) \text{ for some } x \in X.$$

Note: The interval $[0, 1]$ and its square $[0, 1]^2$ are considered with the standard Euclidean distance.

Ex. 5. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Show that $f[\mathbb{R}^2 \setminus S]$ is an interval (possibly improper) for every countable set $S \subset \mathbb{R}^2$.

Ex. 6. Recall, that a topological space is zero-dimensional provided it has a basis formed by clopen (i.e., simultaneously closed and open) sets. Show that every countable normal topological space X is zero-dimensional.

NAME (print): ____________________________

Topology Ph.D. Entrance Exam, August 2012

Write a solution of each exercise on a separate page.

Ex. 1. Let $\langle X, \tau \rangle$ be a topological space and let $\{A_t\}_{t \in T}$ be an indexed family of arbitrary subsets of X. Determine each of the following statements by either proving it or providing a counterexample, where $\mathrm{cl}(A)$ stands for the closure of a set A.

(a) $\bigcap_{t \in T} \mathrm{cl}(A_t) \subset \mathrm{cl}\left(\bigcap_{t \in T} A_t\right)$

(b) $\mathrm{cl}\left(\bigcap_{t \in T} A_t\right) \subset \bigcap_{t \in T} \mathrm{cl}(A_t)$

Ex. 2. Show that a continuous image of a separable space is separable, that is, if there exists a continuous function from a separable topological space X *onto* a topological space Y, then Y is separable. Include the definition of a separable topological space.

Ex. 3. Let f be a continuous function from a compact Hausdorff topological space X into a Hausdorff topological space Y. Consider $X \times Y$ with the product topology. Show that the map $h\colon X \to X \times Y$ given by the formula $h(x) = \langle x, f(x) \rangle$ is a homeomorphic embedding.

Ex. 4. For the topologies τ and σ on $\mathbb{R}$ let symbol $C(\tau, \sigma)$ stand for the family of all continuous functions from $\langle \mathbb{R}, \tau \rangle$ into $\langle \mathbb{R}, \sigma \rangle$.

Let $\mathcal{T}_s$ be the standard topology on $\mathbb{R}$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on $\mathbb{R}$ such that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$. Show that:

(i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$, that is, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(ii) $\mathcal{T}_2 \neq \{\emptyset, \mathbb{R}\}$, that is, $\mathcal{T}_2$ is not trivial.

(iii) $\langle \mathbb{R}, \mathcal{T}_1 \rangle$ is connected.

(Notice that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$ does not imply that either of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ must be equal to the standard topology $\mathcal{T}_s$.)

Ex. 5. Consider the following subsets, $\vdash$ and $\models$, of $\mathbb{R}^2$, where $\mathbb{R}^2$ is endowed with the standard topology:

$$\vdash = (\{0\} \times [-2, 2]) \cup ([0, 2] \times \{0\}) \quad \& \quad \models = (\{0\} \times [-2, 2]) \cup ([0, 2] \times \{-1, 1\}).$$

Prove, or disprove the following:

(i) There exists a continuous function from $\vdash$ onto $\models$.

(ii) There exists a continuous function from $\models$ onto $\vdash$.

Your argument must be precise, but no great details are necessary.

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, August 2013

Solve the following five exercises. Write a solution of each exercise on a separate page. In what follows the symbols $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and A' stand for the interior, closure, and the set of limit points of A, respectively.

Ex. 1. Prove, or disprove by an example, that each of the following properties holds for every subset A of a topological space X.

(a) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

(b) $(A')' = A'$.

Ex. 2. A topological space is a T_0-space provided for every distinct $x, y \in X$ there exists an open set U in X which contains precisely one of the points x and y. Show that X is a T_0-space if, and only if, $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$ for all distinct $x, y \in X$.

Ex. 3. Let $\{A_s : s \in \mathbb{R}\}$ be a family of connected subsets of a topological space X. Assume that $A_s \cap A_t \neq \emptyset$ for every $s, t \in \mathbb{R}$. Show that $A = \bigcup_{s\in\mathbb{R}} A_s$ is connected. (Note, that we do *not* assume that $\bigcap_{s\in\mathbb{R}} A_s \neq \emptyset$.)

Ex. 4. Let $\langle X, d\rangle$ be a metric space and let $A \subset X$ be such that it has no limit points in X, that is, such that $A' = \emptyset$. Show that there exists a family $\{U_a\}_{a\in A}$ of pairwise disjoint open sets such that $a \in U_a$ for every $a \in A$.

Ex. 5. Let X be completely regular; let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f : X \to [0, 1]$ such that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$. (Note, that we do *not* assume that X is normal.)

NAME (print): ____________________

Topology Ph.D. Entrance Exam, May 2015

Solve the following five exercises. Write a solution of each exercise on a separate page. In what follows the symbol int(A) stands for the interior of A. Any subset of $\mathbb{R}$ is considered with the standard topology.

Ex. 1. Let $\langle A_i \rangle_{i=1}^{\infty}$ be an arbitrary sequence of subsets of a topological space X. Show that for any natural number k we have

$$\operatorname{int}\left(\bigcap_{i=1}^{\infty} A_i\right) = \left(\bigcap_{i=1}^{k} \operatorname{int}(A_1)\right) \cap \operatorname{int}\left(\bigcap_{i=k+1}^{\infty} A_i\right).$$

Ex. 2. Let X be a Hausdorff topological space. Show that for every compact subset B of X and any $a \in X \setminus B$ there exist disjoint sets U and V open in X such that $a \in U$ and $B \subset V$. *Do not assume that X is regular!*

Ex. 3. Prove or give a counterexample: The product of two path-connected spaces is also path-connected.

Ex. 4. Let X be an arbitrary topological space and let $\mathbb{Z}$ stand for the set of all integers. Let $\{A_k : k \in \mathbb{Z}\}$ be a family of connected subsets of X. Show that if $A_k \cap A_{k+1} \neq \emptyset$ for every $k \in \mathbb{Z}$, then $\bigcup_{k \in \mathbb{Z}} A_k$ is a connected subset of X.

Ex. 5. Let X be a compact topological space and let $f : X \to \mathbb{R}$ be an arbitrary, **not necessary continuous**, function. Assume that f is locally bounded, that is, that for every $x \in X$ there exists an open $U \ni x$ such that $f[U]$ is bounded in $\mathbb{R}$. Show that $f[X]$ is bounded in $\mathbb{R}$.

NAME (print): ________________________________

Topology Ph.D. Entrance Exam, April 2016

Solve the following five exercises. Write a solution of each exercise on a separate page. In what follows the symbols $\text{int}(A)$ and $\text{cl}(A)$ stand, respectively, for the interior and the closure of A. Any subset of $\mathbb{R}$ is considered with the standard topology, unless stated otherwise.

Ex. 1. Prove, or disprove by giving a counterexample, each of the following statements.

(i) The product of two regular spaces is a regular space.

(ii) The product of two normal spaces is a normal space.

Ex. 2. Prove, directly from the definition, that a compact Hausdorff space is regular. Include the definitions of Hausdorff and regular regular topological spaces.

Ex. 3. Prove that $[0, 1]$, considered with the standard topology, is compact. You can use, without a proof, the standard facts on the order of $\mathbb{R}$.

Ex. 4. Is a continuous image of a separable space separable? Prove it, or give a counterexample. Include a definition of separable topological space.

Ex. 5. Consider $\mathbb{R}^n$, $n \geq 1$, with the standard metric.

(i) Show that an open subset U of $\mathbb{R}^n$ is connected if, and only if, it is path connected. **Hint.** Fix an $x \in U$ and show that the following set $\{y \in U\colon$ there is a path in U from x to $y\}$ is both closed and open in U.

(ii) Give an example of a closed subset F of $\mathbb{R}^n$ which is connected but not path connected.

NAME (print): ______________________

Topology Ph.D. Entrance Exam, August 2007

Solve the following five exercises.

In what follows for a topological space X and $A \subset X$ the symbol $\mathrm{cl}(A)$ will stand for the closure of A.

Ex. 1. Let X and Y be arbitrary topological spaces and consider $X \times Y$ with the product topology. Let $\pi: X \times Y \to X$ be the projection onto the first coordinate, that is, given by $\pi(x, y) = x$.

(a) Show that if Y is compact Hausdorff, then $\pi[F]$ is a closed subset of X for every closed subset F of $X \times Y$.

(b) Give an example of metric spaces X and Y for which the above assertion is false, that is, such that $\pi[F]$ is not closed in X for some closed subset F of $X \times Y$. (Of course, Y cannot be compact in this case.)

Ex. 2. Let X and Y be Hausdorff topological spaces. Let $f: X \to Y$ be an arbitrary function. Show that f is continuous if and only if its graph $G(f) = \{\langle x, f(x) \rangle : x \in X\}$ is a closed subset of $X \times Y$.

Ex. 3. Prove that any infinite normal Hausdorff connected topological space X must be uncountable.

Ex. 4. Let X and Y be topological spaces with $f : X \to Y$ a functions. Show that the following properties are equivalent:

(1) f is continuous.

(2) For every $A \subset X$, $f(\mathrm{cl}(A)) \subset \mathrm{cl}(f(A))$.

(3) For every closed set B in Y, the set $f^{-1}(B)$ is closed.

Ex. 5. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of a topological space X with $A_n \cap A_{n+1} \neq \phi$ for all n. Prove that $\bigcup_{n=1}^{\infty} A_n$ is connected or give a counterexample.

NAME (print): ____________________

Topology Ph.D. Entrance Exam, April 2008

Solve the following five exercises. In what follows for a topological space X and $A \subset X$ the symbol $\mathrm{cl}(A)$ will stand for the closure of A.

Ex. 1. Show that any compact, Hausdorff space is normal.

Ex. 2.

(a) Give a definition of a connected topological space.

(b) Give a definition of a path connected topological space.

(c) Prove that if an open subspace of the plane (considered with the standard topology) is connected, then it is also path connected.

(d) Give an example of a connected subspace of the plane which is not path connected. Indicate, without a formal proof, why your example has the required properties.

Ex. 3. Consider the following four characters as subsets of the plane. Which pairs are are homeomorphic? For non-homeomorphic pairs give the reason.

Characters: B, E, O, T. (Ignore font ornamentation.)

Ex. 4. Let X be a regular topological space and let $\mathcal{F}$ be a family of arbitrary subsets of X. Show that if $\mathcal{F}$ has the following property (N), then so does the family $\mathcal{G} = \{\mathrm{cl}(F) \colon F \in \mathcal{F}\}$.

(N) For every open set U in X and every $x \in U$ there exists an $F \in \mathcal{F}$ such that $x \in F \subset U$.

Ex. 5. Let $\langle X, d\rangle$ be a metric space. Recall that a subspace Y of X is *totally bounded* provided for every $\varepsilon > 0$ the family $\{B(y,\varepsilon) \colon y \in Y\}$ of open ε-balls has a finite subcover of Y. Assume that $Y \subset X$ is totally bounded. Show that the closure $\mathrm{cl}(Y)$ of Y is also totally bounded.

NAME (print): ______________________

Topology Ph.D. Entrance Exam, August 2008

Solve the following four exercises. In what follows the set of real numbers $\mathbb{R}$ as well as any of its subsets is endowed with the natural topology.

Ex. 1. Let X and Y be connected. Show that $X \times Y$ is connected.

Ex. 2. Consider the following result, which you can assume to be true without a proof.

($*$) For every uncountable closed subset F of $\mathbb{R}$ there exists a continuous function from F *onto* the unit interval $I = [0, 1]$.

Which of the following variations of ($*$) is true? Justify your answers.

(a) For every uncountable closed subset F of $\mathbb{R}$ there exists a continuous function from F *onto* $\mathbb{R}$.

(b) For every uncountable closed subset F of $\mathbb{R}$ there exists a continuous function from F *onto* the unit square $I^2 = [0, 1]^2$.

(c) For every uncountable closed subset F of $\mathbb{R}$ there exists a continuous function from F *onto* the Cantor ternary set C. (Recall that Cantor ternary set is obtained by consecutive removal of the middle third open subintervals first from I, then from the remaining two intervals, etc.)

Ex. 3. For a non-empty set J consider $I^J = [0, 1]^J$ with the standard product topology and for every $j \in J$ let $\pi_j \colon I^J \to I$ be the projection onto jth coordinate, that is, defined as $\pi_j(x) = x(j)$ for every $x \in I^J$.

(a) Prove that for every non-empty J and every topological space X a function $f \colon X \to I^J$ if continuous if and only if for every $j \in J$ the mapping $\pi_j \circ f$ is continuous.

(b) Use part (a) to prove that for every metric space X there is a non-empty set J and a continuous *one-to-one* mapping from X into I^J.

Ex. 4. Let $f : X \to X$ be continuous.

(a) Show that if $X = [0, 1]$, then there exists a point $x \in X$ such that $f(x) = x$.

(b) Does this hold if $X = (0, 1]$?

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, April 2009

Solve the following four exercises.

Ex. 1. Let $\{A_i\}_{i=1}^{\infty}$ be a collection of connected subsets of a topological space X such that $A_j \cap A_{j+1} \neq \emptyset$ for all $j = 1, 2, 3, \ldots$. Show that $Y = \bigcup_{i=1}^{\infty} A_i$ is connected, when considered as a subspace of X.

Ex. 2. Let X be a Hausdorff space and let $\{Y_j : j \in J\}$ be an arbitrary family of subsets of X endowed with a subspace topology. Let $Y = \prod_{j \in J} Y_J$ have a product topology. Show that $\Delta = \{y \in Y : y(j) = y(i) \text{ for all } i, j \in J\}$ is a closed subset of Y.

Ex. 3. For a metric space $\langle X, d \rangle$ and non-empty subsets A and B of X define a distance $\text{dist}(A, B) = \inf\{d(a, b) : a \in A \ \& \ b \in B\}$.

(a) Let $\langle X, d \rangle$ be a compact metric space and let $A, B \subset X$ be non-empty and closed. Show that there exist $a \in A$ and $b \in B$ such that $\text{dist}(A, B) = d(a, b)$.

(b) Show, by giving an example, that the conclusion of (a) may be false for a complete metric space $\langle X, d \rangle$ which is not compact.

Ex. 4. Recall that a topological space X is Lindelöf provided every open cover of X has a countable subcover. Show that if $f : X \to Y$ is continuous and X is Lindelöf, then the image $Z = f[X]$ of X is also is Lindelöf, when Z is considered as a subspace of Y.

NAME (print): ____________________

Topology Ph.D. Entrance Exam, Fall 2009

Solve the following four exercises. In what follows the set of real numbers $\mathbb{R}$ as well as any of its subsets is endowed with the natural topology.

Ex. 1.

(a) Prove or give a counterexample to the following statement:

> ($*$) If X is a connected topological space, then for every nonempty proper subset A of X, the boundary of A is nonempty.

(b) Does the converse of ($*$) hold? Prove or give a counterexample.

Ex. 2. Let X be a compact connected Hausdorff space consisting of more than one point. Show that X is uncountable.
Hint: Construct an appropriate function $f\colon X \to \mathbb{R}$.

Ex. 3. Let $A \subset X$, let $f\colon A \to Y$ be continuous, and Y be Hausdorff. Let $\bar{A}$ stand for the closure of A in X.

(a) Show that if f can be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

(b) Give an example of X, Y, and f for which there is no continuous function g as in (a).

(c) Show that the continuous function g as in (a) always exist if X and Y are complete metric spaces and function f is *uniformly* continuous.

Ex. 4. Let $f\colon X \to Y$ be continuous, X is compact, X and Y are Hausdorff, $A \subset X$, and $B = f[A]$. Which of the following statements are true?
When the statement is true, give a short argument for it. (Cite appropriate standard theorem from which it follows and indicate how this imply the result.) When the statement is false, provide a counterexample for it.

(a) If A is closed in X, then so is B in Y.

(b) If A is open in X, then so is B in Y.

(c) If A is connected in X, then so is B in Y.

(d) If A is separable (i.e., it has a countable dense subset) in X, then so is B in Y.

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, April 2010

Solve the following five exercises.

Ex. 1. Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be a metric spaces and let $f: X \to Y$. Show that the following two conditions (different formulations of continuity of f) are equivalent. Include the definition of the limit. In your argument do not use, without a proof, any other characterization of the continuity of f.

(a) $f^{-1}(V)$ is open in X for every open set V in Y.

(b) For every sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X converging to an $x \in X$ we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Ex. 2. Let f be a continuous function from X onto Y. Decide on each of the following statements whether it is true of false. If true, give a short argument for it. If false, provide a counterexample.

(i) If X separable (i.e., has a countable dense set), then so is Y.

(ii) If X metrizable, then so is Y.

(iii) If X has a dense connected subset, then Y is connected.

Ex. 3. Let X be a Hausdorff space. Show that if K is a compact subset of X and $x \in X \setminus K$, then there are open disjoint sets $U \subset X$ and $V \subset X$ such that $x \in U$ and $K \subset V$.

Ex. 4. We say that a space X has a property (H) provided for every distinct $x, y \in X$ there is a continuous function $f: X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$. Let $X = \prod_{n=1}^{\infty} X_n$ be considered with the product topology. Show that if each space X_n has property (H), then so does X.

Ex. 5. For $\varepsilon > 0$ a metric space $\langle X, d\rangle$ is *ε-chain connected* provided for arbitrary $x, y \in X$ there exists a finite sequence $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that $d(x_i, x_{i-1}) \leq \varepsilon$ for all $i = 1, \ldots, n$.

(a) Show that every connected metric space is ε-chain connected for every $\varepsilon > 0$. **Hint.** *Show that the set $\{y \in X\colon y$ is ε-chain connected to $x\}$ is open and closed in X.*

(b) If a metric space $\langle X, d\rangle$ is ε-chain connected for every $\varepsilon > 0$, must it be connected? Prove it or give a counterexample.

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, September 2010

Solve the following five exercises.

In what follows symbols $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ stand for the interior and the closure of a set S, respectively. The n-dimensional Euclidean space $\mathbb{R}^n$ is endowed with the natural topology.

Ex. 1. Let $\langle X, \tau \rangle$ be a topological space and let $\{A_t\}_{t \in T}$ be an indexed family of arbitrary subsets of X. Determine each of the following statements by either proving it or providing a counterexample.

(a) $\bigcup_{t \in T} \mathrm{int}(A_t) \subset \mathrm{int}\left(\bigcup_{t \in T} A_t\right)$

(b) $\mathrm{int}\left(\bigcup_{t \in T} A_t\right) \subset \bigcup_{t \in T} \mathrm{int}(A_t)$

Ex. 2. Let X be a topological space and A be a connected subset of X. Show that the closure of A is connected.

Ex. 3. Let $S^2 = \{\langle x, y, z \rangle \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $f \colon S^2 \to \mathbb{R}$ be continuous. Show that there are two points $p = \langle x, y, z \rangle$ and $p' = \langle x', y', z' \rangle$ on S^2 such that $f(p) = f(p')$ and $x + x' = y + y' = z + z' = 0$.

Ex. 4. Let $\langle X, \tau \rangle$ be a compact Hausdorff space. Show that if $f \colon X \to X$ is continuous, then its graph $G_f = \{\langle x, f(x) \rangle : x \in X\}$ is a closed subset in $X \times X$.

Ex. 5. A topological space $\langle X, \tau \rangle$ is called *collectionwise normal* if for every locally discrete family $\{F_t\}_{t \in T}$ of closed subsets of X there exists a pairwise disjoint family of open sets $\{U_t\}_{t \in T}$ such that $F_t \subset U_t$ for every $t \in T$. (Recall that a family of subsets of subsets of X is called locally discrete when every point of X has a neighborhood that intersects at most one of the sets from the family.) Show that every metric space is collectionwise normal.

Topics to be covered on a PhD entrance exam in topology, Spring 2000

- Examples of topological spaces.
- Separation axioms (T_0-, T_1-, Hausdorff, regular, and normal spaces).
- Metric space topology (completeness, equivalent forms of compactness).
- Continuity.
- Connected spaces.
- Compactness.

Suggested reference books.

- Dugundji, *Topology*, Allyn & Bacon. (Chapters I-IX and XI.)
- Kelly, *General Topology*, D. van Nostrand. (Chapters: all except II, VI, and Appendix.)
- Gemignani, *Elementary Topology*, Addison-Wesley. (Chapters: all except XI.)

NAME (print): ____________________

Topology Ph.D. Entrance Exam, August 2000

In the exercises that follow $\overline{A}$ stands for the closure of A, and $A \setminus B$ for the set difference: $A \setminus B = \{x \in A \colon x \notin B\}$.

Ex. 1. (a) Define a T_0 topological space.

(b) Show that a topological space X is a T_0-space if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ for every distinct $x, y \in X$.

Ex. 2. A topological space X is said to be *completely regular* provided that for each $p \in X$ and closed set A in X such that $p \notin A$, there is a continuous function $f \colon X \to [0,1]$ such that $f(p) = 0$ and $f[A] = \{1\}$.

Prove that any subspace of a completely regular space is completely regular.

Ex. 3. Let X be a topological space and let A and B be non-empty proper closed subsets of X such that $X = A \cup B$. Show that $X \setminus (A \cap B)$ is not connected.

Ex. 4. (a) Give an example of sets A_i $(i = 1, 2, 3, \ldots)$ in a topological space for which

$$\overline{\bigcup_{i=1}^{\infty} A_i} \neq \bigcup_{i=1}^{\infty} \overline{A_i}.$$

(b) Show that for any family $\{A_i \colon i = 1, 2, 3, \ldots\}$ of subsets of a topological space X the following formula holds:

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i} \cup \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} A_i}.$$

Ex. 5. Let $S = \langle \mathbb{R}, \tau_S \rangle$ be a Sorgenfrey line, $D(\mathbb{N}) = \langle \mathbb{N}, \tau_D \rangle$ be a discrete topology on $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $D(\mathbb{R}) = \langle \mathbb{R}, \tau_D \rangle$ be a discrete topology on $\mathbb{R}$.

Show that there is a continuous mapping from S *onto* $D(\mathbb{N})$ but that there is no continuous mapping from S *onto* $D(\mathbb{R})$.

Ex. 6. Let X be a normal space and let U_1 and U_2 be open subsets of X such that $X = U_1 \cup U_2$. Show that there are open sets V_1 and V_2 such that $\overline{V_1} \subset U_1$, $\overline{V_2} \subset U_2$, and $X = V_1 \cup V_2$.

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, April 2001

Ex. 1. Let $\langle X_0, \tau_0\rangle$ and $\langle X_1, \tau_1\rangle$ be connected topological spaces. Show that $X_0 \times X_1$ with the product topology is connected.

Ex. 2. Consider the real line $\mathbb{R}$ with the topology τ generated by the family of intervals:

$$\mathcal{F} = \{[a, b)\colon a \in \mathbb{Q} \ \&\ b \in \mathbb{R} \ \&\ a < b\},$$

where $\mathbb{Q}$ stands for the set of rational numbers. Let X be the product of $\langle \mathbb{R}, \tau\rangle$ with itself (with the product topology). Prove or disprove that X is normal.

Ex. 3. Prove or find a counterexample for the statement:

> A compact subset of a topological space $\langle X, \tau\rangle$ is closed in X.

Ex. 4. Let τ be the usual topology on the real line $\mathbb{R}$. Answer one of the following two questions.

(a) Does there exists a topology $\tau_0 \subset \tau$ such that $\langle \mathbb{R}, \tau_0\rangle$ is homeomorphic to figure eight (i.e., two circles tangent at a point)?

(b) Does there exists a topology $\tau_0 \subset \tau$ such that $\langle \mathbb{R}, \tau_0\rangle$ is homeomorphic to the unit circle $S^1 = \{\langle x, y\rangle \in \mathbb{R}^2\colon x^2 + y^2 = 1\}$?

Ex. 5. Let $\langle X, \tau\rangle$ and $\langle Y, \tau'\rangle$ be the topological spaces and let $f\colon X \to Y$ be a function. Consider the graph $G(f) = \{\langle x, f(x)\rangle\colon x \in X\}$ of f as a subspace of the cartesian product $X \times Y$ (with the product topology). Prove or disprove each the following.

(a) If f is continuous, then $G(f)$ is homeomorphic to X.

(b) If $G(f)$ is homeomorphic to X, then f is continuous.

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Topology Ph.D. Entrance Exam, August 2001

Ex. 1. Let $\mathbb{R}^2$ be the euclidean plane (i.e., with natural topology). Let

$$X = \{\langle x, y\rangle \in \mathbb{R}^2 \colon x^2 + y^2 = 1\} \cup \{\langle x, 0\rangle \in \mathbb{R}^2 \colon -1 \le x \le 1\},$$

$$Y = \{\langle x, y\rangle \in \mathbb{R}^2 \colon x^2 + y^2 = 1\} \cup \{\langle x, 0\rangle \in \mathbb{R}^2 \colon -1 \le x \le 2\}.$$

Are X and Y homeomorphic? Give reasons for your answer.

Ex. 2. Prove that every compact metric space has a countable base for its topology.

Ex. 3. Let $\langle X, d\rangle$ be a compact metric space, and let $f \colon X \to X$ satisfy

$$d(f(x_1), f(x_2)) < d(x_1, x_2) \text{ for all distinct } x_1, x_2 \in X.$$

Show that there is a point $p \in X$ such that $f(p) = p$.

Ex. 4. A topological space X is said to have *countable pseudo character* provided every singleton in X is a G_δ-set (i.e., it is a countable intersection of open sets). Show that every compact Hausdorff space with countable pseudo character is first countable, that is, it has a countable local base at every point $x \in X$.

Ex. 5. Let $\mathcal{F}$ be the family of all *non-zero* polynomials of the form

$$w(x, y) = a_0x^2 + a_1y^2 + a_2xy + a_3x + a_4y + a_5$$

with rational coefficients and for every $w \in \mathcal{F}$. Let

$$E_w = \{\langle x, y\rangle \in \mathbb{R}^2 \colon w(x, y) = 0\}.$$

Show that the plane $\mathbb{R}^2$ is not covered by the sets E_w with $w \in \mathcal{F}$, that is, that $\mathbb{R}^2 \ne \bigcup_{w\in\mathcal{F}} E_w$.

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Topology Ph.D. Entrance Exam
Fall 2004

(1) Let $(\mathbb{R}, T)$ be the Sorgenfrey line (i.e. the set of the real numbers, $\mathbb{R}$, with the topology generated by $\{[a, b)|a, b \in \mathbb{R} \text{ with } a < b\}$).
   (a) Prove or disprove: $(\mathbb{R}, T)$ is homeomorphic to $\mathbb{R}$ in the standard topology generated by $\{(a, b)|a, b \in \mathbb{R} \text{ with } a < b\}$.

   (b) Show that the Sorgenfrey line is first countable, but not metrizable.

(2) Let (X, T) be a topological space, and let (Y, T') be a Hausdorff space. For a function $f : X \to Y$, let $G(f)$ denote the graph of f; specifically,

$$G(f) = \{(x, f(x)) \in X \times Y : x \in X\}$$

where $X \times Y$ is the Cartesian product with the product topology.
   (a) Prove that if $f : X \to Y$ is continuous, then $G(f)$ is a closed subset of $X \times Y$.

   (b) Give an example of a function f from the reals $\mathbb{R}^1$ (standard topology) to the reals $\mathbb{R}^1$ (standard topology) such that the graph $G(f)$ is closed in the plane $\mathbb{R}^1 \times \mathbb{R}^1$ but such that f is not continuous.

(3) Show that $\mathbb{R}$ and $\mathbb{R}^n$ (in the usual topologies) are not homeomorphic if $n > 1$.

(4) The Cartesian product of any two connected topological spaces is connected.

(5) Answer T for true, F for false. Indicate reasons for your answers without details.
   (a) ☐ The Cartesian product of any two normal Hausdorff spaces is normal.

   (b) ☐ Every topological space is a continuous image of a metric space.

   (c) ☐ Every Hausdorff space is regular.

   (d) ☐ The intersection of two connected metric spaces is connected.

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Topology Ph.D. Entrance Exam
Spring 2005

Choose four (and only four) of the following problems and work them neatly on each page on which the problem is stated. You will be provided with scratch paper for calculation and preliminary work, but this should not be turned in. All work to be graded should be written up in readable form on the paper provided.

(1) Prove that if (X, T_1) and (Y, T_2) are connected topological spaces, then $X \times Y$ with the product topology T is connected.

(2) Let (X, T) be a Hausdorff space, and let $p_1, p_2, \ldots, p_n$ be finitely many distinct points of X. Prove that there are open subsets $U_1, U_2, \ldots, U_n$ of X such that $p_i \in U_i$ for all i and $U_i \cap U_j = \phi$ for all $i \neq j$.

(3) Let S be the Sorgenfrey topology for the real line $\mathbb{R}$ (i.e., S is generated by all intervals for the form $[a, b)$ for $a, b \in \mathbb{R}$). Let $\mathbb{Q}$ denote the rational numbers with their usual topology T (i.e. $\mathbb{Q}$ has the subspace topology determined by the usual topology on $\mathbb{R}$).

(a) Is there a continuous function from $(\mathbb{R}, S)$ onto $(\mathbb{Q}, T)$? Justify your answer.

(b) Is there a continuous function from $(\mathbb{R}, S)$ onto $\mathbb{R}$ with the usual topology? Justify your answer.

(4) Let (X, d) be a compact metric space, and let $\mathcal{U}$ be an open cover of X. Show that there exists $\epsilon > 0$ such that for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that the open ball $B_d(x, \epsilon) \subset U_x$.

(5) Let (X, d_1) and (Y, d_2) be metric spaces with their usual topologies T_{d_1} and T_{d_2}, respectively. Let $p \in X$ and let $f : X \to Y$ be a function. Prove that f is continuous at p if and only if f is sequentially continuous at p (i.e., for each sequence $\{s_n\}_{n=1}^{\infty}$ converging to p, the sequence $\{f(s_n)\}_{n=1}^{\infty}$ converges to $f(p)$).

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Topology Ph.D. Entrance Exam

Fall 2005

Work each individual problem including subparts and choose one from each of the following grouped problems neatly on each page on which the problem is stated. Work only three problems total. You will be provided with scratch paper for calculation and preliminary work, but this should not be turned in. All work to be graded should be written up in readable form on the paper provided.

Group 1 (choose one)

(1) Let (X, T) be a topological space, and let $A \subset X$. Let χ_A denote the characteristic function for A (i.e., $\chi_A(x) = 0$ if $x \in X - A$ and $\chi_A(x) = 1$ if $x \in A$). What conditions on A are both necessary and sufficient for χ_A to be continuous ($\{0, 1\}$ has the usual topology)? Prove your answer is correct.

(2) Let X be a metric space with its usual topology T.
   (a) Prove that if (X, T) is separable, then every collection of mutually disjoint open subsets of X is countable.
   (b) Is the converse of (a) true?

Group 2 (choose one)

(3) Let $Y \subset X$ and let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

(4) Show that if X is normal, then the members of any pair of disjoint closed sets have neighborhoods whose closures are disjoint.

(5) Let (X, T) be a topological space, let Y be a set, and let f be a function from X onto Y. Let

$$T_f = \{U \subset Y : f^{-1}(U) \in T\}.$$

(a) Prove that T_f is a topology.

(b) Let (Z, T_Z) be a topological space, and let $g : Y \to X$ be a function. Prove that if the composition $g \circ f$ is $T - T_Z$ continuous, then g is $T_f - T_Z$ continuous.

(c) Let X be the interval $[0, 2\pi]$ with the usual topology T, and let Y be the set whose members are the following subsets of X : $\{x\}$ if $0 < x < 2\pi$ and $\{0, 2\pi\}$. Let $f : X \to Y$ be the natural map (i.e., $f(x)$ is the member of Y containing x). Let T_f be as in (a). What familiar space do you think (Y, T_f) is? Verify your answer (hint: make use of (b)).