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Cycle double covers and non-separating cycles

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ABSTRACT

Which 2-regular subgraph R of a cubic graph G can be extended to a cycle double cover of G? We provide a condition which ensures that every R satisfying this condition is part of a cycle double cover of G. As one consequence, we prove that every 2-connected cubic graph which has a decomposition into a spanning tree and a 2-regular subgraph C consisting of k circuits with $k \leq 3$, has a cycle double cover containing C.

1. Introduction and definitions

All graphs in this paper are assumed to be finite. A *trivial* component is a component consisting of one single vertex. In the context of cycle double covers the following definitions are convenient. A *circuit* is a 2-regular connected graph and a *cycle* is a graph such that every vertex has even degree. Thus every 2-regular subgraph of a cubic graph is a cycle.

In this paper the following concept is essential: a subgraph C of a connected graph H is called *non-separating* if $H - E(C)$ is connected, and *separating* if $H - E(C)$ is disconnected. Hence, every non-separating cycle C in a connected cubic graph H with $|V(H)| > 2$ is an induced subgraph of H if C does not have a trivial component.

A *cycle double cover (CDC)* of a graph G is a set S of cycles such that every edge of G is contained in the edge sets of precisely two elements of S. The well known *Cycle Double Cover Conjecture (CDCC)* ([15,17–19]; or see [23]) states that every bridgeless graph has a CDC. It is known that the CDCC can be reduced to snarks, i.e. cyclically 4-edge connected cubic graphs of girth at least 5 admitting no 3-edge coloring, see for instance [23]. There are several versions of the CDCC, see [23]. The subsequent one by Seymour is called the *Strong-CDCC* (see [6,7], or, see Conjecture 1.5.1 in [23]) and it is one of the most active approaches to the CDCC.

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Conjecture 1.1. *Let G be a bridgeless graph and let C be a circuit of G. Then G has a CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Note that the Strong-CDCC cannot be modified by replacing "circuit" with "cycle" since there are infinitely many snarks which would serve as counterexamples, see [4,11]. For instance, the Petersen graph P_{10} has a 2-factor, C_2 say, but P_{10} does not have a CDC $\mathcal{S}$ such that $C_2 \in \mathcal{S}$. We underline that C_2 is separating! Here we only consider CDCs of graphs containing prescribed non-separating cycles. In particular the following conjecture by the first author has been a motivation for this paper.

Conjecture 1.2 ([20]). *Let C be a non-separating cycle of a 2-edge connected cubic graph G. Then G has a CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Recall that a *decomposition* of a graph G is a set of edge-disjoint subgraphs covering $E(G)$. Hence, if a connected cubic graph G has a decomposition into a tree T and a cycle C, then C is a non-separating cycle of G. Note that all snarks with less than 38 vertices have a decomposition into a tree and a cycle and that there are infinitely many snarks with such a decomposition, see [14]. We consider the following equivalent reformulation of the above conjecture (see Proposition 1.5).

Conjecture 1.3. *Let G be a 2-edge connected cubic graph which has a decomposition into a tree T and a cycle C. Then G has a CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Our main result is the following.

Theorem 1.4. *Let G be a 2-edge connected graph with a decomposition into a tree T and a cycle C with $k \leq 3$ components. Then G has a CDC $\mathcal{S}$ with $C \in \mathcal{S}$ and in particular the following holds. (1) If $k \leq 2$, then G has a 5-CDC $\mathcal{S}_2$ with $C \in \mathcal{S}_2$. (2) Let $k = 3$. Then G has a 5-CDC $\mathcal{S}_3$ with $C \in \mathcal{S}_3$ if G is not contractible to the Petersen graph; otherwise G has a 6-CDC $\mathcal{S}'_3$ with $C \in \mathcal{S}'_3$.*

Theorem 1.4 shows that Conjecture 1.3 is true if the cycle C has at most three components. Note that Theorem 1.4 is valid for all 2-edge connected graphs. The proof is based on Theorem 3.1 and results which imply the existence of nowhere-zero 4-flows. Graphs constructed from the Petersen graph demand special treatment in the proof, see Theorem 1.4 (2). In Section 4 we consider applications of Theorem 1.4 and Theorem 3.1 for cubic graphs. In Section 5 we present some remarks and one more conjecture.

Note that the tree T in Conjecture 1.3 is a *hist* (see [1]), that is a spanning tree without a vertex of degree two (hist is an abbreviation for *h*omeomorphically *i*rreducible *s*panning *t*ree). Conversely, every cubic graph with a hist has trivially a decomposition into a tree and a cycle. For informations and examples of snarks with hists, see [13,14]. Let us also mention that Conjecture 1.3 limited to snarks is stated in [14].

Proposition 1.5. *Conjectures 1.2 and 1.3 are equivalent.*

Proof. Obviously, it suffices to show that the truth of Conjecture 1.3 implies the truth of Conjecture 1.2. Suppose that C is a non-separating cycle of a 2-edge connected cubic graph G such that the graph $G_C := G - E(C)$ is not a tree. Let T_C be a spanning tree of G_C. Then the non-trivial components of $G_C - E(T_C)$ can be paths or circuits and all are non-separating in G_C. Denote by X the edge set

$$\{e \in E(G_C - E(T_C)) \ : \ \text{e is not contained in a circuit of } G_C - E(T_C)\} .$$

Denote by Y_1 the maximal 2-regular subgraph of $G_C - E(T_C)$ which may be empty. Now, subdivide in G each of the edges of X two times and add an edge joining these two new vertices to obtain a circuit of length two and call the union of these circuits of length two Y_2. Thus we obtain a new cubic graph G' and it is straightforward to see that G' has a hist T' such that the 2-regular subgraph of $G' - E(T')$ denoted by C' consists of $Y_1 \cup Y_2 \cup C$. Obviously every CDC of G' containing C' corresponds to a CDC of G containing C. □

For terminology not defined here, we refer to [3]. For more informations on cycle double covers and flows, see [22,23].

2. Preliminary/lemmas

If v is a vertex of a graph then we denote by E_v the set of edges incident with v. A *k-CDC* of a graph G is a set $\mathcal{S}$ of k cycles of G such that every edge of G is contained in the edge sets of precisely two elements of $\mathcal{S}$. In our understanding of a cycle C, $E(C) = \emptyset$ is possible.

Lemma 2.1 (*Goddyn [9] and Zhang [21], or see [22] Lemma 3.5.6*). *Let G be a graph admitting a nowhere-zero 4-flow and let C be a cycle of G. Then G has a 4-CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

The following Lemma is well known and can easily be proved by using a popular result of Tutte, namely that a graph has a nowhere-zero k-flow if and only if it has a nowhere-zero $\mathbb{Z}_k$-flow.

Lemma 2.2. *Let G be a graph and C be a subgraph of G such that $G/E(C)$ has a nowhere-zero k-flow. Then G admits a k-flow f with $supp(f) \supseteq E(G) - E(C)$.*

Definition 2.3. Let G and H be two graphs. Then G is called (k, H)-girth-degenerate if and only if there are a sequence of graphs $G_0 = G, G_1, \ldots, G_m$ and a sequence of circuits $C_0, C_1, \ldots, C_{m-1}$ such that

(1) $C_i \subseteq G_i$ and $|E(C_i)| \le k$ for $i = 0, 1, 2, \ldots, m-1$,

(2) $G_{i+1} = G_i/E(C_i)$ for $i = 0, 1, 2, \ldots, m-1$ and

(3) $G_m = H$.

Moreover, we call G in short k-girth-degenerate if G is (k, K_1)-girth-degenerate.

Note that we consider a loop as a circuit of length one and that loops can arise in the course of contractions. For instance every complete graph is 3-girth-degenerate and every 2-connected planar graph is 5-girth-degenerate. Note also that H in the above definition is a special minor of G and that $m = 0$ implies G is (k, G)-girth-degenerate.

Lemma 2.4 (*Catlin [5], or, see Lemma 3.8.11 of [22], p. 80*). *Let G be a graph and let $C \subseteq G$ be a circuit of length at most 4. If $G/E(C)$ admits a nowhere-zero 4-flow, then so does G.*

Lemma 2.5. *Every 4-girth-degenerate graph G admits a nowhere-zero 4-flow.*

Proof. Apply induction on the number of contractions to obtain K_1 (see Definition 2.3) and apply Lemma 2.4. □

3. Main results

Every theorem in this section has been motivated by questions on cubic graphs and was first stated for them. Nevertheless, cubic graphs are not mentioned in the theorems presented here since the original results were generalized.

Theorem 3.1. *Let G be a 2-edge connected graph. Suppose that C is a non-separating cycle of G such that $G/E(C)$ has a nowhere-zero 4-flow. Then G has a 5-CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Proof. Since $G/E(C)$ has a nowhere-zero 4-flow, G has by Lemma 2.2 a 4-flow f such that $supp(f) \supseteq E(G) - E(C)$. Set $E_0 = \{e : f(e) = 0\}$. Obviously, $E_0 \subseteq E(C)$. Since $G - E(C)$ is connected, there is a circuit C_e of $G - (E(C) - \{e\})$ containing e. Set $J_1 = \triangle_{e \in E_0} C_e$. Then J_1 contains every edge of E_0 but no edge of $C - E_0$. Moreover, set $J_2 = C \triangle J_1$. Then J_2 is a cycle contained in $supp(f)$ which contains all edges of $C - E_0$. Since $G - E_0$ has a nowhere-zero 4-flow, there is by Lemma 2.1 a 4-CDC $\mathcal{S}_1$ of $G - E_0$ with $J_2 \in \mathcal{S}_1$. Then the set $\mathcal{S} = (\mathcal{S}_1 - \{J_2\}) \cup \{J_1, C\}$ is a 5-CDC of G with $G \in \mathcal{S}$. □

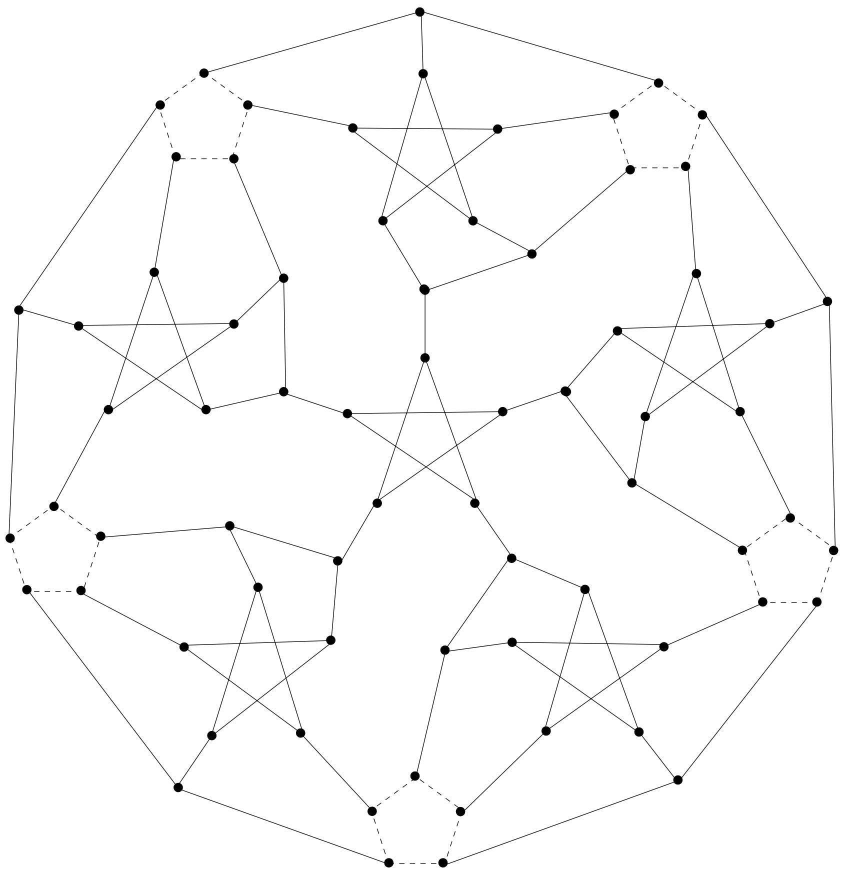

Fig. 1. A snark Q^* with a non-separating cycle C^* illustrated by dashed edges.

Note that Theorems 3.1 and 3.2 are equivalent statements. Theorem 3.1 follows from Theorem 3.2 since E_0 (see the proof of Theorem 3.1) defines M and thus Theorem 3.2 can be applied. The converse direction is shown in the proof of Theorem 3.2. Note also that E_0, respectively, M is a matching if G is cubic in Theorem 3.1, respectively, Theorem 3.2.

Theorem 3.2. *Let G be a 2-edge connected graph which contains a non-separating cycle C. Suppose that G has an edge subset $M \subseteq E(C)$ such that $G - M$ has a nowhere-zero 4-flow. Then G has a 5-CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Proof. Since $G - M$ has a nowhere-zero 4-flow and because $M \subseteq E(C)$, $G/E(C)$ has a nowhere zero 4-flow. By applying Theorem 3.1, the result follows. □

Note that we cannot prove Theorem 1.4 directly via Theorem 3.1. Consider for instance the cubic graph, Q say, which results from P_{10} by expanding each u_1, u_2, u_3 to a triangle, see Fig. 2. Then Q has a decomposition into a tree and a cycle C with three components consisting of triangles. Moreover, $Q/E(C)$ does not have a nowhere-zero 4-flow and thus Theorem 3.1 cannot be applied. Note also that C is not contained in a 5-CDC of Q.

We proceed in our preparation for the proof of Theorem 1.4. To keep the proof of Theorem 1.4 short, we next prove several specials results.

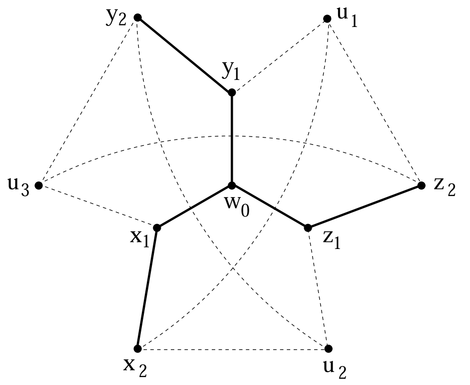

Fig. 2. Illustration of G_n and U_n where $G_n \cong P_{10}$ (T is shown in bold face).

Proposition 3.3. *Let G be a 2-edge connected graph with a vertex subset U such that*

$$G - U \text{ is acyclic and } d_G(v) > 2 \text{ for every } v \in V(G) - U \qquad (*)$$

Suppose $|U| \leq 3$. *Then*

(1) G is 2-girth-degenerate if $|U| = 1$,

(2) G is 4-girth-degenerate if $|U| = 2$, *and*

(3) G is 4-girth-degenerate or $(4, P_{10})$*-girth-degenerate if* $|U| = 3$.

Proof. Let (G, U) be a pair such that G is a 2-edge connected graph and $U \subseteq V(G)$. Suppose that (G, U) satisfies condition $(*)$. If there exists a circuit $C \subseteq G$ with $|E(C)| \leq 4$, then we call C a *small circuit* of (G, U) and we set $G' := G/E(C)$ and $U' := \{v_C\} \cup \{U - V(C)\}$ where v_C is the vertex in G' obtained from contracting C (note that $V(C) \cap U \neq \emptyset$ since $G - U$ is acyclic by hypothesis). We call (G', U') a *small contraction* of (G, U) and observe that $|U'| \leq |U|$. Furthermore, we call a sequence of pairs $\{(G_i, U_i)\}_{i=1}^n$ a *small contraction sequence* if (G_{i+1}, U_{i+1}) is a small contraction of (G_i, U_i) for each $i = 1, \ldots, n-1$. It is clear that if (G_1, U_1) satisfies condition $(*)$, then every pair (G_i, U_i) satisfies condition $(*)$ for $i = 2, \ldots, n$, and G_i is 2-edge connected if $i < n$. Note that $|U_1| \geq \cdots \geq |U_n|$ holds and that U_i may equal $V(G_i)$ for some i.

For a given (G, U), let $\{(G_i, U_i)\}_{i=1}^n$ be a maximal small contraction sequence with $(G_1, U_1) = (G, U)$. Hence there is no small circuit of (G_n, U_n) since the sequence is maximal. In particular, there is no parallel edge with one end in U. Denote by $\hat{N}_{U_n}(v)$ the neighbors of v of G_n lying in U_n. We say two leaf-vertices v_1 and v_2 of $V(G_n) - U_n$ are a *bad pair* if $|\hat{N}_{U_n}(v_1) \cap \hat{N}_{U_n}(v_2)| \geq 2$. It is evident that there is no bad pair in (G_n, U_n), otherwise one can easily find a 4-circuit by using the bad pair and two common neighbors of them.

Before we use all of the introduced concepts, we prove the first part of the proposition.

Proof of (1) It suffices to prove that every (G, U) with $|U| = 1$ contains a 2-circuit intersecting U since we then can proceed by induction. Obviously, every component, say T, of $G - U$ is a tree. Since $d_G(v) > 2$ (see condition $(*)$) for every leaf-vertex $v \in V(T)$, v is adjacent via a parallel edge to $u \in U$ and thus G contains the desired 2-circuit.

Proof of (2) We proceed by contradiction. So, let $S := \{(G_i, U_i)\}_{i=1}^n$ be a maximal small contraction sequence with $(G_1, U_1) = (G, U)$ and suppose that $G_n \neq K_1$. $G_n - U_n = \emptyset$ would imply that there is a small 2-circuit or a 1-circuit which contradicts the maximality of S. Thus, there is a component T of $G_n - U_n$. T is not a single vertex otherwise there will be a pair of parallel edges incident with a vertex of U_n. Hence T contains two leaf-vertices. If $|U_n| = 1$, then each of them is adjacent via a parallel edge to $u \in U$, a contradiction. If $|U_n| = 2$, then the two leaf-vertices of T form a bad pair, a contradiction.

Proof of (3) Let $S := \{(G_i, U_i)\}_{i=1}^{n}$ be a maximal small contraction sequence with $(G_1, U_1) = (G, U)$ and suppose that $G_n \neq K_1$. We show that $G_n \cong P_{10}$ which will prove statement (3). Since $|U_1| \geq \cdots \geq |U_n|$ and since statements (1) and (2) above hold, $|U_n| = 3$. Call a vertex subset $W \subseteq V(G_n) - U_n$ a *bad set* if $d_{G_n - U_n}(w_1, w_2) \leq 2$ for any $w_1, w_2 \in W$ and $\sum_{w \in W} |\hat{N}_{U_n}(w)| \geq 4$. Suppose that G_n has a bad set W. The latter inequality and $|U_n| = 3$ imply that one vertex of U has two neighbors in W and the distance condition implies that G_n has a small circuit, a contradiction. Hence, G_n does not have a bad set.

Obviously, $G_n - U_n \neq \emptyset$ otherwise $G_n[U_n]$ contains a small circuit. Suppose that $G_n - U_n$ has two components H_1 and H_2. Recall that G_n does not contain a small circuit. If H_1 consists of a single vertex h_1, then one can find a vertex h_2 in H_2 such that h_1, h_2 form a bad pair. If neither H_1 nor H_2 is a single vertex, then each contains two leaf-vertices. Hence there are four leaf-vertices and each has a pair of distinct (since G_n does not contain a 2-circuit) neighbors in U_n. Since U_n can provide at most three different pairs of these neighbors, two of the leaf-vertices form a bad pair by Pigeonhole principle. Therefore $G_n - U_n$ is connected and thus a tree which we denote by T. T is not a single vertex otherwise there will be a small 3-circuit since G_n is 2-edge connected. Moreover, T cannot have exactly two leaves since then T will be a path $v_0 v_1 \cdots v_k$, and thus either $\{v_0, v_1, v_2\}$ forms a bad set if $k \geq 2$ or $\{v_0, v_1\}$ forms a bad set if $k = 1$. Indeed, T cannot have four or more leaves, otherwise one can choose a bad pair from these leaves by Pigeonhole principle. Therefore T has exactly three leaves and thus there is a unique degree 3-vertex, say w_0. Hence T consists of three edge disjoint paths: $w_0 x_1 \cdots x_j$, $w_0 y_1 \cdots y_k$, $w_0 z_1 \cdots z_l$ with $j, k, l \geq 1$. We claim that $j = k = l = 2$. If one of $\{j, k, l\}$, say $j > 2$, then $\{x_j, x_{j-1}, x_{j-2}\}$ forms a bad set. If one of $\{j, k, l\}$, say $k = 1$, then $\{x_1, y_1, z_1\}$ will also form a bad set. Hence $j = k = l = 2$. Since there is no bad pair, by symmetry, we may assume that $\hat{N}_{U_n}(x_2) = \{u_1, u_2\}$, $\hat{N}_{U_n}(y_2) = \{u_2, u_3\}$, $\hat{N}_{U_n}(z_2) = \{u_3, u_1\}$ where $U_n = \{u_1, u_2, u_3\}$, see Fig. 2. Since G_n does not have a small circuit, we must also have $x_1 u_3, y_1 u_1, z_1 u_2 \in E(G_n)$. Then G_n is isomorphic to P_{10}. $\square$

Lemma 3.4. *Let G be a graph with a non-separating cycle C of G. Suppose $G/E(C)$ is $(4, H)$-girth-degenerate where H is a graph admitting a k-CDC and satisfies $\Delta(H) \leq 3$. Then the following holds.*

(1) G has a $(k+1)$-CDC $\mathcal{S}$ with $C \in \mathcal{S}$ if $k \geq 5$.

(2) G has a 5-CDC $\mathcal{S}$ with $C \in \mathcal{S}$ if $k \leq 4$.

Proof. Let G_3 be a 2-edge connected graph having a nontrivial edge-cut E_s with $|E_s| = s$, $s \in \{2, 3\}$ such that $G_3 - E_s$ consists of two components G_1 and G_2. Define two new graphs $\hat{G}_1 := G_3/E(G_2)$ and $\hat{G}_2 := G_3/E(G_1)$. Denote the unique vertex in $\hat{G}_1$ ($\hat{G}_2$) which has been obtained from contracting $E(G_2)$ ($E(G_1)$) by g_2 (g_1). Let $C_i \subseteq \hat{G}_i$, $i = 1, 2$ be a cycle such that $g_1 \notin V(C_2)$ and $g_2 \notin V(C_1)$. Then C_1, C_2 are cycles of G_3 and $C_1 \cup C_2$ is also a cycle of G_3. The following fact can be verified straightforwardly and will be used in the end of the proof.

Fact 3.5. *Let $\hat{G}_i$, $i = 1, 2$ have a k_i-CDC $\mathcal{S}_i$ with $C_i \in \mathcal{S}_i$ and suppose $k_1 \leq k_2$. Then G_3 has a k_2-CDC $\mathcal{S}_3$ with $C_1 \cup C_2 \in \mathcal{S}_3$.*

If $|V(H)| = 1$, then $G/E(C)$ has a nowhere-zero 4-flow by Lemma 2.5 and thus Lemma 3.4 follows by applying Theorem 3.1. Hence we assume $|V(H)| > 1$.

Call a vertex $w_0 \in V(H)$ *big* if it corresponds to a subgraph W_0 of G with $|V(W_0)| > 1$, i.e. W_0 is connected and $E_{w_0} \subseteq E(H)$ corresponds to an s-edge-cut E_s of G for some $s \in \{2, 3\}$ such that one component of $G - E_s$ is W_0 (note that $\Delta(H) \leq 3$ by hypothesis). Thus $E(C) \cap E_s = \emptyset$. Therefore every component of C is either a subgraph of W_0 or disjoint with W_0.

We prove the lemma by induction on the number of big vertices of H denoted by $b(H)$. If $b(H) = 0$, then $G = H$ and thus $E(C) = \emptyset$ and the lemma holds. Now suppose $b(H) = n + 1$. Let w_0 be a big vertex of H and let W_0 be its corresponding subgraph in G.

Define the graph $J := G/E(W_0)$ and the cycle $C_J \subseteq J$ induced by its edge set $E(C_J) := E(C) - (E(C) \cap E(W_0))$. Then C_J is non-separating in J. Moreover,

$$J/E(C_J) = G/(E(C) \cup E(W_0)).$$

Recall that $G/E(C)$ is $(4, H)$-girth-degenerate and observe that no circuit is contracted which intersects E_{w_0}, respectively, E_s in order to obtain H from $G/E(C)$. Thus, by the above equation and since $G/E(C)$ is $(4, H)$-girth-degenerate, $J/E(C_J)$ is $(4, H)$-girth-degenerate.

Furthermore, H has a k-CDC by assumption. Therefore all conditions of the considered lemma are fulfilled and since H has (with respect to J) precisely n big vertices, J has a CDC $\mathcal{S}_J$ with $C_J \in \mathcal{S}_J$ satisfying statements (1), (2) (if we replace G by J, S by S_J, and C by C_J).

To obtain the desired CDC of G, define the graph $J' := G/E(G - V(W_0))$ (recall that W_0 is connected) and denote the unique vertex of J' which is not part of W_0 by x. Let $C_{J'} \subseteq J'$ be the cycle induced by its edge set $E(C_{J'}) := E(C) - E(C_J)$. Since $G/E(C)$ is $(4, H)$-girth-degenerate and w_0 a big vertex, it follows that $W_0/E(C)$ is 4-girth-degenerate and thus $J'/E(C_{J'})$ is 4-girth-degenerate. Hence $J'/E(C_{J'})$ has a nowhere-zero 4-flow by Lemma 2.4. Since $C_{J'}$ is a non-separating cycle of J', there is by Theorem 3.1 a 5-CDC $\mathcal{S}_{J'}$ of J' with $C_{J'} \in \mathcal{S}_{J'}$.

Depending on the value of k (concerning the k-CDC of H) there are two cases.

Case 1. $k \geq 5$. Then $\mathcal{S}_J$ is a $(k+1)$-CDC of J. Since $\mathcal{S}_{J'}$ is a 5-CDC of J', and $k+1 > 5$, Fact 3.5 implies that $C = C_{J'} \cup C_J$ is contained in a $(k+1)$-CDC $\mathcal{S}$ of G (note that $x \notin V(C_{J'})$ and that $w_0 \notin V(C_J)$).

Case 2. $k \leq 4$. Then $\mathcal{S}_J$ is a 5-CDC of J and $\mathcal{S}_{J'}$ is a 5-CDC of J'. Fact 3.5 implies that $C = C_{J'} \cup C_J$ is contained in a 5-CDC $\mathcal{S}$ of G. □

Definition 3.6. Let H be a graph and $v \in V(H)$ with $av, bv \in E_v$. Then we say that the graph $(H - av - bv) \cup ab$ is obtained from H by splitting away the edges av and bv.

Proof of Theorem 1.4. It is straightforward to see that we can assume that G does not have a vertex of degree two. Moreover, we can also assume that $V(C) \subseteq V(T)$. If $V(C) \not\subseteq V(T)$, we form from G and C a new graph $\hat{G}$ (without changing the tree T) and a new cycle $\hat{C} \subseteq \hat{G}$ (having again k components). Regard each component C_i, $i \in \{1, \ldots, k\}$ of C as an eulerian closed trail. For every vertex $v \in V(C_i)$ in G with $d_{C_i}(v) \geq 4$ satisfying $v \notin V(T)$, we split repeatedly pairs of consecutive edges (of the trail) having both v as endvertex, away, until T becomes a spanning tree and we denote this obtained cycle by $\hat{C}$. It is straightforward to verify that every r-CDC of $\hat{G}$ which contains $\hat{C}$ corresponds to a r-CDC of G which contains C. Hence we assume $V(C) \subseteq V(T)$.

Since C has at most three components, $G' = G/E(C)$ satisfies the conditions of Proposition 3.3 (replace G by G'). We can assume that G' is not $(4, P_{10})$-girth-degenerate, otherwise we apply Lemma 3.4 to G with $H = P_{10}$ (since P_{10} has a 5-CDC). Thus G' is at most 4-girth-degenerate by Proposition 3.3. By Lemma 2.5, G' admits a nowhere-zero 4-flow. Moreover, C is non-separating since $G - E(C)$ is a tree. Hence the conditions of Theorem 3.1 are fulfilled and its application finishes the proof. □

4. Corollaries for cubic graphs

Within this section we show some applications of Theorems 1.4 and 3.1 for cubic graphs. For this purpose we need the following definition and lemma.

Definition 4.1. An evenly spanning cycle of a graph G is a spanning cycle C of G such that for every component L of C the number of vertices in L with odd degree in G is even.

For instance, $V(G)$ is an evenly spanning cycle of G if G is an eulerian graph. In contrast to the latter example, an evenly spanning cycle of a $2k + 1$-regular graph cannot contain a trivial component. Note that every hamiltonian circuit is an evenly spanning cycle.

Lemma 4.2 ([23] or [2])**.** *The following statements are equivalent:*

(1) A graph G has a nowhere-zero 4-flow. (2) G has an evenly spanning cycle.

Corollary 4.3. *Let G be a 2-edge connected cubic graph. Suppose that C is a non-separating cycle of G such that $G/E(C)$ has a hamiltonian circuit. Then G has a 5-CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Proof. Since a hamiltonian circuit in $G' := G/E(C)$ is an evenly spanning cycle, G' has a nowhere-zero 4-flow by Lemma 4.2. By applying Theorem 3.1, the result follows. □

Corollary 4.4. *Let G be a 2-edge connected cubic graph with a 2-factor consisting of two chordless circuits C_1, C_2. Then G has a 5-CDC $\mathcal{S}$ with $C_1 \in \mathcal{S}$.*

Proof. Since C_1 is non-separating and $G/E(C_1)$ is hamiltonian, the result follows by applying Corollary 4.3. □

Remark 4.5. C_1 in Corollary 4.4 is part of some CDC even if C_1 is allowed to have chords, see [8]. G in Corollary 4.3 has some CDC even if C is separating, see [10]. The above results offer some insight into which cycles are part of a 5-CDC (see the Strong 5-CDCC in [12]).

The next result follows directly from Theorem 1.4.

Corollary 4.6. *Let G be a 2-edge connected cubic graph with a cycle $C \subseteq G$ such that (i) C has at most three components and (ii) $G - E(C)$ is acyclic and has at most two components $\{T_1, T_2\}$.*

Then G has a CDC if $T_k \cup C$ is bridgeless for each $k \in \{1, 2\}$.

Corollary 4.7. *Let G be a 2-edge connected cubic graph which has a decomposition into a spanning tree T, k_1 circuits and k_2 edges such that $k_1 + k_2 \leq 3$. Then G has a CDC containing the cycle consisting of the k_1 circuits.*

Proof. Since the CDCC is known to hold for graphs with small order, we can assume that $k_1 \neq 0$. Subdivide each of the k_2 edges two times and add an edge joining these two vertices to obtain a circuit of length two. Then we obtain a new graph G' with a hist T' for which we can apply Theorem 1.4 since $G' - E(T')$ has $k_1 + k_2 \leq 3$ circuits. Moreover, the CDC of G' corresponds to a CDC of G which contains all k_1 circuits of $G - E(T)$. □

Corollary 4.8. *Every cyclically 4-edge connected cubic graph which has a decomposition into a tree and a cycle C consisting of k circuits with $k \leq 3$ has a 5-CDC $\mathcal{S}$ with $C \in \mathcal{S}$.*

Proof. Every cubic graph which is contractible to P_{10} is either P_{10} itself or a cubic graph with a cyclic 3-edge cut. Since for every decomposition of P_{10} into a tree and a 2-regular subgraph, the 2-regular subgraph consists of one circuit (see [14]), the proof follows by applying Theorem 1.4. □

5. Remarks and open problems

We know that Conjecture 1.2 is not implied by Theorem 3.1 (recall the graph Q defined below the proof of Theorem 3.2). Is this still the case if we restrict Conjecture 1.2 to snarks? The graph Q^* illustrated in Fig. 1 is a snark which has a non-separating cycle C^* (which is contained in a CDC) but Theorem 3.1 is not applicable since $Q^*/E(C^*)$ does not have a nowhere-zero 4-flow. Q^* is constructed from the graph P' in [16, Fig. 12.1]: P' does not admit a nowhere-zero 4-flow and Q^* is obtained from P' by contracting double edges and expanding vertices of degree five to 5-circuits. Observe also that C^* is a maximal non-separating cycle of Q^*, i.e. Q^* does not have a larger non-separating cycle $\hat{C}$ satisfying $C^* \subset \hat{C}$.

With respect to Conjecture 1.3, we do not know a cyclically 4-edge connected cubic graph which prevents the direct application of Theorem 3.1.

Problem 5.1. Does there exist a snark G which has a decomposition into a tree and a cycle C such that $G/E(C)$ does not have a nowhere-zero 4-flow?

The truth of the next conjecture implies the truth of the CDCC and in particular the truth of the 5-CDCC, see Theorem 3.1.

Conjecture 5.2. *Every cyclically* 4*-edge connected cubic graph* G *contains a non-separating cycle* C *such that* $G/E(C)$ *has a nowhere-zero* 4*-flow.*

Note that Conjecture 5.2 would be false if G is not demanded to be cyclically 4-edge connected. For instance, the cyclically 3-edge connected cubic graph which is obtained from K_4 by replacing every vertex of K_4 with a copy of $P_{10} - v,\ v \in V(P_{10})$ would then form a counterexample.

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