European Journal of Combinatorics 96 (2021) 103344

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Berge–Fulkerson coloring for some families of superposition snarks

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ARTICLE INFO

Article history:
Received 16 November 2020
Accepted 30 March 2021
Available online xxxx

ABSTRACT

It is conjectured by Berge and Fulkerson that every bridgeless cubic graph has six perfect matchings such that each edge is contained in exactly two of them. This conjecture has been verified for many families of snarks with small (≤ 5) cyclic edge-connectivity. An infinite family, denoted by $\mathcal{S}_K$, of cyclically 6-edge-connected superposition snarks was constructed in [European J. Combin. 2002] by Kochol. In this paper, the Berge–Fulkerson conjecture is verified for the family $\mathcal{S}_K$, and, furthermore, some larger families containing $\mathcal{S}_K$. This is the first paper about the Berge–Fulkerson conjecture for superposition snarks and cyclically 6-edge-connected snarks. Tutte's integer flow and Catlin's contractible configuration are applied here as the key methods.

1. Introduction

The Berge–Fulkerson conjecture is one of the most famous open problems in graph theory.

Conjecture 1.1 (*Berge–Fulkerson Conjecture, (simply say B-F-Conjecture) [16], or see [34,36]*)**.** *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

The six perfect matchings in the B-F-Conjecture are called a *Fulkerson-cover* (or *B-F-coloring*). The B-F-Conjecture can be restated as an edge coloring problem, that is, for every bridgeless cubic graph G, there exists a proper 6-edge-coloring of the graph $2G$, where the graph $2G$ is obtained from G by duplicating every edge to become a pair of parallel edges.

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https://doi.org/10.1016/j.ejc.2021.103344
0195-6698/

Although the statement of the B-F-Conjecture is very simple, the solution has eluded many mathematicians over five decades and remains beyond the horizon.

The B-F-Conjecture is equivalent to the statement that *every bridgeless cubic graph has a family of six even subgraphs such that every edge is covered precisely four times*. It was proved by Bermond, Jackson and Jaeger [2] that every bridgeless graph has a family of *seven* even subgraphs such that every edge is covered precisely four times. Fan [11] proved that every bridgeless graph has a family of *ten* even subgraphs such that every edge is covered precisely six times.

Similar to other major open problems, such as, Tutte's 5-flow conjecture, cycle double cover conjecture, etc., the B-F-Conjecture is trivial for 3-edge-colorable cubic graphs, and remains widely open for snarks. And it is well-known that snarks play an essential role for the B-F-Conjecture, flow problems, and cycle cover problems in graph theory, see [45].

The relation between Berge–Fulkerson coloring and shortest cycle cover problems was discovered in [12,37]. Some generalizations and variations of the B-F-Conjecture have been proposed [15,34], such as, the 5-perfect matching covering problem (Berge conjecture), k-cycle double cover problems, etc. The structures of graphs with unique Berge–Fulkerson coloring were characterized in [35]. And a necessary and sufficient condition for the conjecture was obtained in [19] (and its integer flow version in [9]).

In [5,7,19,20,30] etc., the B-F-Conjecture was verified for various families of snarks such as Petersen graph, generalized Blanuša snarks, Szekeres snarks and flower snarks, Watkins snarks, Celmins–Swart snarks, Szekeres–Watkins snarks, Goldberg snarks, Isaace snarks, Loupekine snarks, Lukot'ka-Máčajová-Mazák-Škoviera snarks, snarks with order at most 36, Abreu–Labbate–Rizzi–Sheehan snarks and Hägglund–Hoffmann–Ostenhof snarks etc. (See literature [1,3,5,8,17,18,22,29, 38,41–43] etc. for references or surveys of snarks.)

In [24,32,33] etc., Fan–Raspaud conjecture [12], a weaker version of the B-F-Conjecture, was verified partially by Kaiser, Raspaud, Máčajová, Škoviera and Steffen etc.

Note that all these snarks are of small (≤ 5) girth or small (≤ 5) cyclical edge-connectivity. It was shown in [31], a possible minimum counterexample for the B-F-Conjecture should have cyclic edge-connectivity at least 5.

As we known, *superposition*, introduced by Kochol [27], is an effective method to construct infinite family of snarks (also see [13,14] for a similar idea given by Fiol). For example, using superposition methods, Kochol [27] had obtained many snarks without small cycles. He also constructed a family, denoted by $\mathcal{S}_K$, of cyclically 6-edge-connected snarks in [26,28] (call them Kochol snarks). But there was no any result yet about the B-F-Conjecture for these superposition snarks in the literature.

In this paper, the Berge–Fulkerson conjecture is verified for this infinite family $\mathcal{S}_K$ of snarks, see Definitions 2.17 for the constructions of $\mathcal{S}_K$.

Theorem 1.2. *Every Kochol snark is Berge–Fulkerson colorable.*

Actually, the Berge–Fulkerson conjecture can be verified for a larger family of snarks, (see Theorem 4.7 and Corollary 4.8), which contains the family $\mathcal{S}_K$ and is a generalization of the construction in [26,28].

In Section 2, some notations, definitions and necessary lemmas are presented. Our main results and proofs are presented in Section 3. Further extensions and remarks are presented and discussed in Section 4.

The approach of this paper is different from almost all early works. Tutte's integer flow theory is applied here as the frame of the proof, and Catlin's lemma is used as one of the major techniques.

2. Preliminaries

2.1. Notations and terminology

For most standard notation and terminology, we follow Bondy and Murty [4], West [44] and Diestel [10].

A *circuit* is a 2-regular connected subgraph and an *even subgraph* is a graph with even degree at every vertex. For the sake of convenience, a circuit of length k is called a *k-circuit*, and a circuit of length at most k is called a *k^--circuit*.

The *suppressed graph*, denote by $\overline{G}$, is the graph obtained from G by suppressing all degree-2 vertices. A *k-factor* of a graph G is a spanning k-regular subgraph of G. The set of edges of a 1-factor of a graph G is called a *perfect matching* of G.

Let G be a graph, the degree of the vertex v in G is denoted by $d_G(v)$. Denote by $d(u, v)$ the distance between vertices u and v which is the length of a shortest path joining u and v in G. Denote by $[n] = \{1, 2, \ldots, n\}$ and P_{10} the Petersen graph and K_n a complete graph with n vertices.

2.2. Key lemmas for the B-F-conjecture and integer 4-flows

Readers are referred to [45] for definitions and notations about Tutte's flow theory. The following are key lemmas in the proofs of the main theorems in this paper.

Lemma 2.1 (*[19]*). *A cubic graph G is Berge–Fulkerson colorable if and only if there are two edge-disjoint matchings J_1 and J_2 such that*

(1) $J_1 \cup J_2$ is an even subgraph Q in G, and

(2) for each $i \in \{1, 2\}$ and for each component X of $G\backslash J_i$, either the suppressed graph $\overline{X}$ is 3-edge-colorable, or, X is a circuit.

Lemma 2.2 (*Tutte [40]*). *The admission of a nowhere-zero 4-flow is equivalent to the 3-edge-coloring for cubic graphs.*

By Lemma 2.2, Lemma 2.1 was restated in [9] as follows.

Lemma 2.3 (*[9]*). *A cubic graph G is Berge–Fulkerson colorable if and only if there are two edge-disjoint matchings J_1 and J_2 such that*

(1) $J_1 \cup J_2$ is an even subgraph Q in G, and

(2) for each $i \in \{1, 2\}$, the suppressed graph $\overline{G\backslash J_i}$ admits a nowhere-zero 4-flow.

Lemma 2.4 (*Catlin [6], or see Lemma 3.8.11 of [45]*). *Let G be a 2-edge-connected graph containing a circuit C of length at most 4. If the contracted graph G/C admits a nowhere-zero 4-flow, then G also admits a nowhere-zero 4-flow.*

Definition 2.5 (*[21]*). (i) Let G and H be two graphs. Then G is called (k, H)-girth-degenerate if and only if there are a sequence of graphs $G_0 = G, G_1, \ldots, G_m$ and a sequence of circuits $C_0, C_1, \ldots, C_{m-1}$ such that

(1) $C_i \subseteq G_i$ and $|E(C_i)| \leq k$ for $i = 0, 1, 2, \ldots, m-1$,

(2) $G_{i+1} = G_i/E(C_i)$ for $i = 0, 1, 2, \ldots, m-1$ and

(3) $G_m = H$.

(ii) Specifically, a graph G is *k-girth-degenerate* if G is (k, K_1)-girth-degenerate.

The following lemma is a well known corollary of Lemma 2.4.

Lemma 2.6. *Every 4-girth-degenerate graph G admits a nowhere-zero 4-flow.*

Lemma 2.7 (*[21]*). *Let G be a 2-edge connected graph with a vertex set U such that $G - U$ is acyclic and $d_G(v) > 2$ for every $v \in V(G) - U$.*

Suppose $|U| \leq 3$. Then

(1) G is 2-girth-degenerate if $|U| = 1$,

(2) G is 4-girth-degenerate if $|U| = 2$, and

(3) G is 4-girth-degenerate or $(4, P_{10})$-girth-degenerate if $|U| = 3$.

Theorem 2.8 (*Jaeger [23]*). *Every 4-edge-connected graph admits a nowhere-zero 4-flow.*

2.3. Superposition

Notation in this and the next subsections are about the superposition snarks, constructed by Kochol in [25–28].

Definition 2.9 (*Kochol [25–28]*)**.** A *multipole* M is a triple (V, E, S) which consists of a set of vertices $V = V(M)$, a set of edges $E = E(M)$, and a set of semiedges $S = S(M)$ such that each semiedge is incident with either one vertex or another semiedge, in which case the two mutually incident semiedges form a so-called *isolated edge*.

All multipoles in superedge and supervertex considered here are cubic graphs, i.e., any vertex is incident with precisely three edges or semiedges.

Definition 2.10. (i) M is called a *k-pole* if $|S(M)| = k$. If $S(M) = \emptyset$, then M is an ordinary graph and is denoted by $M = (V, E)$ in this case.

(ii) If S is partitioned into pairwise disjoint non-empty sets S_i of order k_i, for $i = 1, 2, \ldots, m$, then the k-pole M is a $(k_1, k_2, \ldots, k_m)$-pole, denoted by $M = (V, E, S_1, S_2, \ldots, S_m)$ for $k = k_1+k_2+\cdots k_m$. Each S_j is called a *connector* of M.

(iii) By a *superedge* we mean any multipole with two connectors and by a *supervertex* we mean any multipole with three connectors.

An example. A multipole with seven semiedges and two isolated edges is shown in Fig. 1, and the three connectors are S_i for $i \in [3]$.

Definition 2.11. Let G be a cubic graph and u_1, u_2 be two non-adjacent vertices of G.

(i) Denote by $(G)_{u_1,u_2}$ a $(3, 3)$-pole (V, E, S_1, S_2) obtained from G by deleting u_1 and u_2 and replacing the edges incident with u_i by semiedges of S_i for $i \in \{1, 2\}$.

(ii) The $(3, 3)$-pole $(G)_{u_1,u_2}$ is a *proper superedge* if G is a snark.

(iii) The vertices u_1, u_2 are called *connector-vertices* of the superedge $(G)_{u_1,u_2}$.

Following [27], we now present the notion of superposition.

Definition 2.12 (*[27,28]*)**.** Let $G = (V, E)$ be a cubic graph, and replace each edge $e \in E$ by a superedge $\mathcal{E}(e)$ and each vertex $v \in V$ by a supervertex $\mathcal{V}(v)$. Assume that if $v \in V$ is incident with $e \in E$, then one connector of $\mathcal{V}(v)$ is accompanied with one connector of $\mathcal{E}(e)$ and that these two connectors have the same cardinality. Join the semiedges of the accompanying pairs of connectors. The resulting cubic graph is a *superposition* of G with $\mathcal{V}$ and $\mathcal{E}$, denoted by $G(\mathcal{V}, \mathcal{E})$. $G(\mathcal{V}, \mathcal{E})$ is a *proper superposition* of G if $\mathcal{E}(e)$ is proper for every $e \in E$.

Remark. Note, it is possible that some edge e is replaced by a "trivial superedge", the edge e itself.

The following lemma ensures the existence of superposition snarks.

Lemma 2.13 (*[27]*)**.** *If G is a snark and $G(\mathcal{V}, \mathcal{E})$ is a proper superposition of G, then $G(\mathcal{V}, \mathcal{E})$ is a snark.*

2.4. Flower snarks, Kochol snarks

The first cyclically 6-edge-connected snark, denoted by G_{118}, was constructed in [26]. The snark G_{118} is further generalized to an infinite family $\mathcal{S}_K$ of cyclically 6-edge-connected snarks in [28]. Each member of $\mathcal{S}_K$ is a superposition in which superedges are obtained from flower snarks (Definition 2.15) and supervertices are constructed in Definition 2.14.

Definition 2.14 (*Supervertices [28]*)**.** There are two types of supervertices used in the construction of snarks of $\mathcal{S}_K$. One of them is A, illustrated in Fig. 1. Another one A', illustrated in Fig. 2, is obtained from A by inserting two vertices x, y into two isolated edges, respectively, and joining the vertices x, y with an edge.

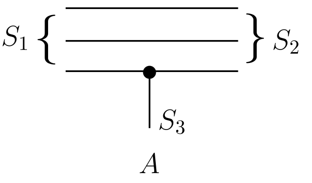

Fig. 1. The multipole A with seven semiedges and two isolated edges and three connectors.

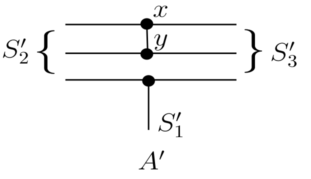

Fig. 2. The multipole A'.

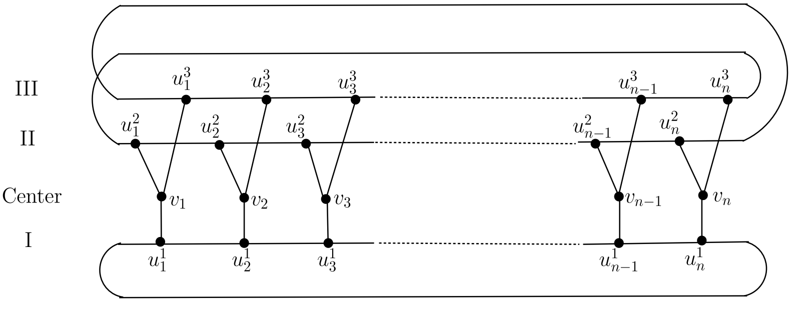

Fig. 3. The first drawing of flower snark F_n.

For odd integer $n \geq 5$, the *flower snark* F_n of order $4n$ is constructed [22] as follows.

Definition 2.15 (*Flower snarks [22]*)**.** Let n be an odd integer at least 5. The flower snark F_n is constructed as follows (see Fig. 3).

(i) The vertex set $V(F_n)$ has a partition $\{V_{\mathrm{I}}, V_{\mathrm{II}}, V_{\mathrm{III}}, V_{\mathrm{C}}\}$ where

$$V_{\mathrm{I}} = \{u_i^1 : \ 1 \leq i \leq n\}, V_{\mathrm{II}} = \{u_i^2 : \ 1 \leq i \leq n\},$$
$$V_{\mathrm{III}} = \{u_i^3 : \ 1 \leq i \leq n\} \text{ and } V_{\mathrm{C}} = \{v_i : \ 1 \leq i \leq n\}.$$

(ii) The graph is comprised of a circuit $u_1^1u_2^1 \cdots u_n^1u_1^1$ of length n and a circuit $u_1^2u_2^2 \cdots u_n^2u_1^3u_2^3 \cdots u_n^3u_1^2$ of length $2n$, and in addition,

(iii) each vertex v_i ($i \in [n]$) is adjacent to u_i^1, u_i^2 and u_i^3.

Remark. Figs. 3 and 4 are two classical drawings of the flower snarks.

Definition 2.16 (*Kochol Flower Superedges*)**.** Let F_n be a flower snark given in Definition 2.15, and u, v be two vertices with $d(u, v) \geq 3$. Let $(F_n)_{u,v}$ be a (3,3)-pole obtained from F_n by deleting the vertices u and v and retaining the resulting semiedges.

(i) [26,28] A set $\mathcal{B}_K$ of superedges is constructed as follows.

$$\mathcal{B}_K = \{(F_n)_{u,v} : \ u \in V_{\mathrm{I}}, v \in V_{\mathrm{C}}, d(u, v) \geq 3 \text{ for } n = 2k+1 \text{ and } k \geq 2\},$$

(see Fig. 5 for $n = 5$, $u = v_1$ and $v = u_3^1$).

(ii) Superedges constructed in (i) are called <u>*Kochol flower superedges*</u>.

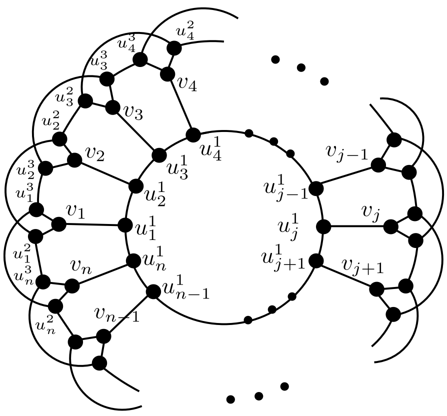

Fig. 4. The second drawing of flower snark F_n.

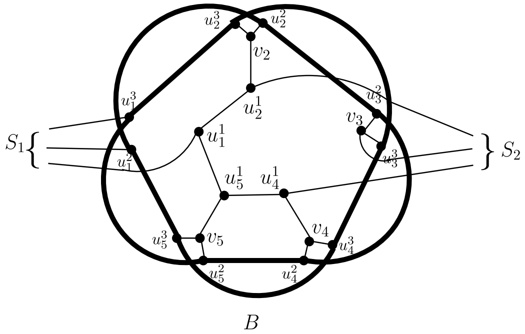

Fig. 5. A multipole B with two connectors S_1 and S_2.

An infinite family $\mathcal{S}_K$ of cyclically 6-edge-connected snarks [28] is constructed as follows.

Definition 2.17 (*[28]*)**.** Let C^* be a 6-circuit of the Petersen graph P_{10}. The set of vertices outside C^* is $\{t_0, t_1, t_2, t_3\}$. Every member of $\mathcal{S}_K$ is constructed from P_{10} by replacing every vertex of C^* with a copy of either A (see Fig. 1) or A' (see Fig. 2), and every edge of C^* with a copy of any member of $\mathcal{B}_K$. Leaving the rest of P_{10} unchanged, and joining the corresponding semiedges of the copies of supervertices and of superedges (see Fig. 6).

The following two multipoles play an important role in the proof of our main results

Definition 2.18. Let M be a (3,3)-pole with connectors S_1 and S_2,

(1) M is called a *star multipole* and denoted by *MS* if it consists of a degree-6 vertex v and all semiedges incident with v (see the left one in Fig. 7).

(2) M is called a *double star multipole* and denoted by *DMS* if it consists of two adjacent vertices u and v such that each connector S_i, for $i \in \{1, 2\}$, has two semiedges incident with u, and one semiedge incident with v (see the right one in Fig. 7).

2.5. Some technical lemmas

The following is a technical lemma that will be used in the proof of the main results.

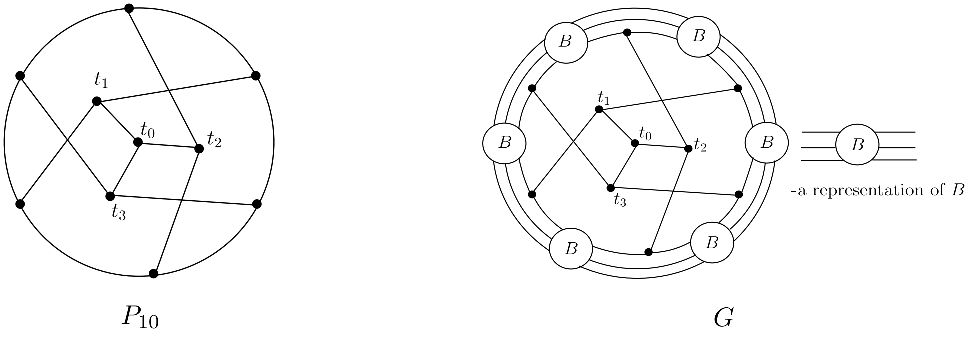

Fig. 6. The Petersen graph and the graph G of $\mathcal{S}_K$.

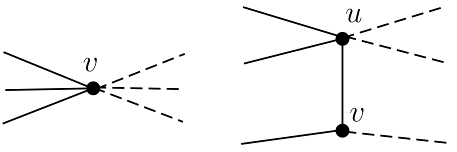

Fig. 7. *MS* and *DMS*. (Legends: solid-lines are S_1-semiedges and dashed-lines are S_2-semiedges.)

Lemma 2.19. *Let G be a 2-edge-connected loopless graph with n $(1 < n \leq 7)$ vertices and m edges. Further assume that G is simple when $n \geq 6$. If*

$$m \geq \begin{cases} n+1, & \text{if } n \leq 6; \\ n+2, & \text{if } n = 7, \end{cases}$$

then G is 4-girth-degenerate.

Proof. Let G be a counterexample to the lemma with $|E(G)|$ as small as possible.

Claim 2.19.1. *The girth of G is at least* 5.

Proof. Otherwise, a smaller counterexample could be obtained by contracting a 4^--circuit. ■

By Claim 2.19.1, G is simple.

Claim 2.19.2. *Every circuit of G is chordless and G has no Hamilton circuit.*

Proof. Since $n \leq 7$, any circuit with a chord must contain a circuit of length ≤ 4. This contradicts Claim 2.19.1. Since $m \geq n+1$, every Hamilton circuit (if exists) must have a chord. ■

Claim 2.19.3. $n = 7$.

Proof. By Claim 2.19.1, $n \geq 5$. Let C be a longest circuit of G.

If $n = 5$, then, by Claim 2.19.1, C must be a Hamilton circuit. This contradicts Claim 2.19.2.

If $n = 6$, then, by Claims 2.19.1 and 2.19.2, C is of length 5. Since $n = 6$, let $\{w\} = V(G) - V(C)$. Note that w has two neighbors in $V(C)$. Thus, G contains a circuit of length at most 4 consisting of w and a shorter segment of C. This contradicts Claim 2.19.1. ■

The final step. By Claims 2.19.1 and 2.19.2, $|V(C)| = 5, 6$.

Case 1. $|C| = 6$. Since $m \geq 9$ and C is chordless, the only vertex w outside C is adjacent to three vertices of C. Then G must contain a circuit of length at most 4 consisting of w and a shorter segment of C. This contradicts Claim 2.19.1.

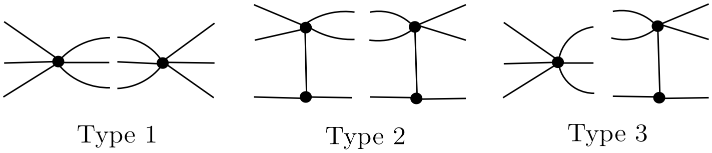

Fig. 8. Three types.

Case 2. $|C| = 5$. Let $C = v_1 \cdots v_5 v_1$ and $\{w_1, w_2\} = V(G) - V(C)$. Similar to Case 1, $|N(w_i) \cap V(C)| = 1$ for every $w_i \notin V(C)$, and, therefore, $w_1 w_2 \in E(G)$. Without loss of generality, G contains a path $P = v_1 w_1 w_2 v_3$ (by avoiding shorter circuit). Note that $v_1 w_1 w_2 v_3 v_4 v_5 v_1$ is a circuit of length 6. This contradicts that C is longest. □

3. Main theorems

Lemma 3.1 is the key lemma in the proof of main results. Its lengthy proof is presented in Section 3.2.

Lemma 3.1. *If $B \in \mathcal{B}_K$ (a Kochol flower-superedge in Definition 2.16), then*
(1) B contains an even subgraph $Q = J_1 \cup J_2$ with J_i being a matching,
(2) for each $i \in \{1, 2\}$, $\overline{B \backslash J_i}$ is either $(4, MS)$-girth-degenerate or $(4, DMS)$-girth-degenerate. (See Definitions 2.18 and 2.5.)

3.1. *Proof of Theorem 1.2*

Assume Lemma 3.1 holds, we will prove Theorem 1.2.

Note that $C^* = x_1 x_2 \cdots x_6 x_1$ is a 6-circuit of the Petersen graph P_{10} and $U = V(P_{10}) \backslash V(C^*)$. By Definition 2.17, the superposition snark G is constructed from P_{10} by replacing every edge $x_\mu x_{\mu+1} \in E(C^*)$ with a superedge $B^\mu \in \mathcal{B}_K$ and every vertex $x_\nu \in V(C^*)$ with one of $\{A, A'\}$ for each $\mu, \nu \in [6]$. (See Definitions 2.14 and 2.16 for $\mathcal{B}_K$ and $\{A, A'\}$ respectively.)

By Lemma 3.1, each B^μ contains an even subgraph Q^μ such that

(1) Q^μ is the union of two matchings J_1^μ and J_2^μ and

(2) $\overline{B^\mu \backslash J_i^\mu}$ is either $(4, MS)$-girth-degenerate or $(4, DMS)$-girth-degenerate.

Denote

$$Q^+ = \bigcup_{\mu=1}^{6} Q^\mu, \quad J_1^+ = \bigcup_{\mu=1}^{6} J_1^\mu, \text{ and } J_2^+ = \bigcup_{\mu=1}^{6} J_2^\mu.$$

In the remaining part of the proof, we are to show that $\overline{G \backslash J_i^+}$, for each $i \in \{1, 2\}$, is 4-girth-degenerate. Thus, by Lemma 2.6 and Lemma 2.3, G admits a B-F-coloring.

Claim 3.1.1. *For each $i \in \{1, 2\}$, $(\overline{G \backslash J_i^+}) \backslash U$ is 4-girth-degenerate.*

Proof. Recall that $U = V(P_{10}) \backslash V(C^*)$, and, for each $i \in \{1, 2\}$, $(\overline{G \backslash J_i^+}) \backslash U$ is the subgraph obtained from the 6-circuit C^* by replacing each edge $x_\mu x_{\mu+1}$ with $\overline{B^\mu \backslash J_i^\mu}$ and each vertex x_ν with $\mathcal{V}(x_\nu) \in \{A, A'\}$.

By Lemma 3.1, the graph $\overline{B^\mu \backslash J_i^\mu}$ is either $(4, MS)$-girth-degenerate or $(4, DMS)$-girth-degenerate. That is, by recursively contracting a sequence of circuits, say $\mathcal{X}_i^\mu$, it becomes one of $\{MS, DMS\}$. Denote the resulting subgraph by H_i^μ which contains 1 or 2 vertices.

There are three types of pairs $\{H_i^\mu, H_i^{\mu+1}\}$ (illustrated in Fig. 8). Let $\mathcal{V}(x_{\mu+1})$ be the supervertex between the superedges B^μ and $B^{\mu+1}$. Let L_i be the graph obtained from $(\overline{G \backslash J_i^+}) \backslash U$ by contracting

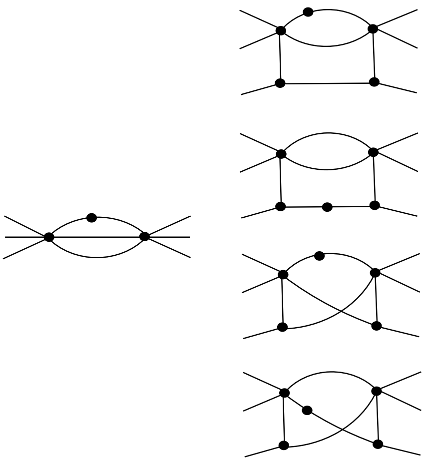

Fig. 9. Seven types with supervertex A.

$\bigcup_{\mu=1}^{6} \mathcal{X}_i^{\mu}$. The subgraph $H_i^{(\mu,\mu+1)}$ of L_i induced by vertices of H_i^{μ}, $H_i^{\mu+1}$ and $\mathcal{V}(x_{\mu+1})$ contains at most 7 vertices and satisfies all conditions of Lemma 2.19. (See Figs. 9 and 10.) By Lemma 2.19, $H_i^{(\mu,\mu+1)}$ is 4-girth-degenerate (by recursively contracting a sequence of circuits, say $\mathcal{Y}_i^{\mu}$), which implies that $(\overline{G\backslash J_i^+})\backslash U$ is 4-girth-degenerate. ■

Claim 3.1.2. *The graph P_{10}/C^*, obtained from Pertersen graph by contracting the 6-circuit C^*, is 4-girth-degenerate.*

Proof. This claim is straightforward since P_{10}/C^* (see Fig. 11) contains 5 vertices and 9 edges and satisfies all conditions of Lemma 2.19. By Lemma 2.19, the graph P_{10}/C^*, after recursively contracting a sequence of circuits, say $\mathcal{Z}$, becomes a single vertex. This completes the proof of the claim. ■

The final step. The combination of the above two claims yields that, for each $i \in \{1, 2\}$, the suppressed graph $\overline{G\backslash J_i^+}$, after recursively contracting the circuits in

$$\bigcup_{\mu\in[6]} \mathcal{X}_i^{\mu} \cup \bigcup_{\mu\in[6]} \mathcal{Y}_i^{\mu} \cup \mathcal{Z},$$

becomes a single vertex.

Thus, by Lemma 2.6, each $\overline{G\backslash J_i^+}$ ($i \in \{1, 2\}$) admits a nowhere-zero 4-flow and, therefore, by Lemma 2.3, G admits a B-F-coloring. □

3.2. Proof of Lemma 3.1

In order to apply Lemma 2.6, we are to find a sequence of 4^--circuits recursively for a processing of repeated contractions as follows. Let $B \in \mathcal{B}_K$ and $\mu \in \{1, 2\}$. We are going to find

(a) an integer m,
(b) a sequence $\mathcal{X}$ of subgraphs $X_0, X_1, \ldots, X_{m-1}$ of $B\backslash J_\mu$ and
(c) a sequence of contracted subgraphs $B_0, \ldots, B_m$ of $B\backslash J_\mu$ such that

(1) $B_0 = \overline{B\backslash J_\mu}$,
(2) for every $i \in \{0, 1, \ldots, m-1\}$,

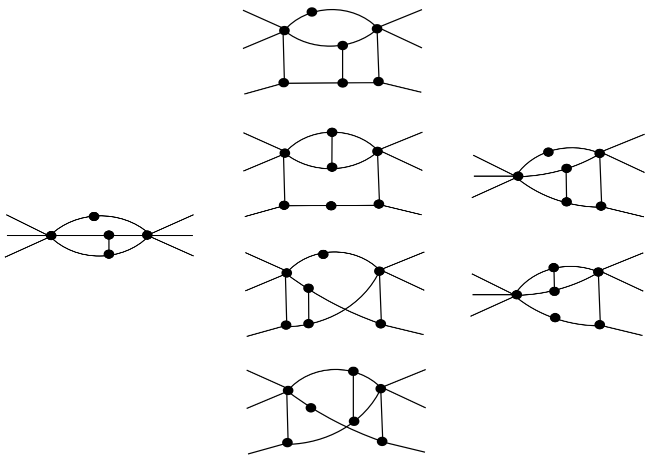

Fig. 10. Seven types with supervertex A'.

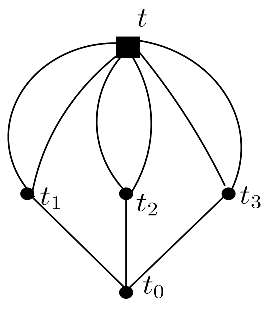

Fig. 11. The graph P_{10}/C^*.

(2-i) each subgraph X_i corresponds to a 4^--circuit or a union of some 4^--circuits in B_i.

(2-ii) $B_{i+1} = B_i/E(X_i)$, and,

(3) B_m is an *MS* or *DMS* (see Fig. 7).

Note that for any $B \in \mathcal{B}_K$, B is a multipole $(F_n)_{uv}$, where u, v are two *connector-vertices* of the flower snark F_n with $d(u, v) \geq 3$, where $u \in V_{\rm I}$, $v \in V_{\rm C}$ and $\{V_I, V_{II}, V_{III}, V_C\}$ is the partition of $V(F_n)$. (See Definition 2.16.)

Consider $B \in \mathcal{B}_K$. That is, $u \in V_I$ and $v \in V_C$.

Without loss of a generality, let $v = v_1$. As $d(u, v) \geq 3$, the vertex u is u_j^1 for $3 \leq j \leq n - 1$. By symmetry of the flower snark, it is sufficient to consider

$$3 \leq j \leq \frac{n+1}{2}. \tag{1}$$

Case 1. Consider the flower snark F_n with $n = 5$. The inequality (1) implies that $j = 3$.

Let

$$Q = u_2^3 u_3^3 u_4^3 v_4 u_4^2 u_3^2 u_2^2 v_2 u_2^3,$$

and J_1, J_2 be a pair of edge-disjoint perfect matchings of Q. (See Fig. 12, in which bold-lines are J_1-edges and dashed-lines are J_2-edges.)

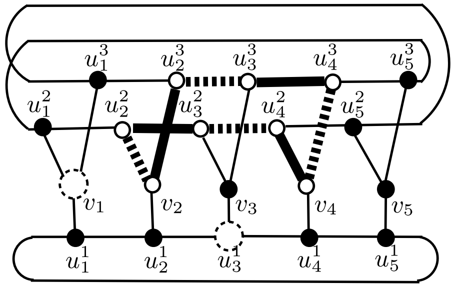

Fig. 12. Case 1: $n = 5$ and $j = 3$. (Legends: bold-lines are J_1-edges, dashed-lines are J_2-edges and thin-lines are edges not in Q. Dotted-circles v_1 and u_3^1 are two connector-vertices, hollow-circles are vertices of Q and solid-circles are vertices not in Q.)

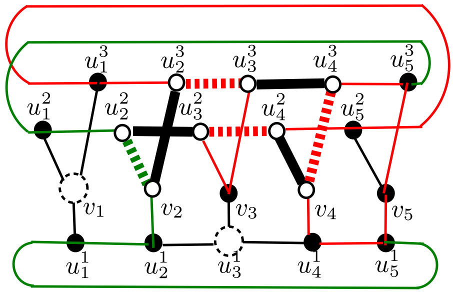

Fig. 13. Case 1: $B \backslash J_1$ with $n = 5$ and $j = 3$. (Legends: $E(X_0)$ is red, $E(X_1) - E(X_0)$ is green.) (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Let $\pi : V(F_n) \to V(F_n)$ such that

$$\pi(x) = \begin{cases} x, & \text{if } x \in V_C \cup V_I; \\ u_\mu^2, & \text{if } x = u_\mu^3; \\ u_\mu^3, & \text{if } x = u_\mu^2. \end{cases} \tag{2}$$

Note that π is an automorphism of the flower snark F_n with $\pi(J_1) = J_2$. Thus, $\overline{B \backslash J_1} \cong \overline{B \backslash J_2}$, and, therefore, it is sufficient to consider only $B \backslash J_1$ (see Fig. 13).

Let

$$C_{0,1} = u_5^3 v_5 u_5^1 u_4^1 v_4 u_4^3 u_5^3, \quad C_{0,2} = u_1^3 u_5^2 u_4^2 u_3^2 v_3 u_3^3 u_2^3 u_1^3, \quad C_{1,1} = u_5^3 v_5 u_5^1 u_1^1 u_2^1 v_2 u_2^2 u_1^2 u_5^3,$$

and

$$X_0 = C_{0,1} \cup C_{0,2}, \quad X_1 = C_{1,1}.$$

Claim 3.2.1. *The subgraph X_i corresponds to a union of 4^--circuits in B_i for $i = 0, 1$.*

Proof. Although $C_{0,1}$ is a 6-circuit and $C_{0,2}$ is a 7-circuit in $B \backslash J_1$, each of them corresponds to a 4^--circuit in the suppressed subgraph $B_0 = \overline{B \backslash J_1}$ since $\{v_4, u_4^2, u_4^3, u_3^3, u_3^2, u_2^2, v_2, u_2^3\}$ $(= V(Q))$ is the set of degree-2 vertices of $B \backslash J_1$ (hollow-circles in the figure), each of these degree two vertices is suppressed in B_0.

It is similar for $C_{1,1}$. The circuit $C_{1,1}$ is of length 8 in $B \backslash J_1$ and corresponds to a 4-circuit in the contracted subgraph $B_1 = B_0 / X_0$ since the segment $u_5^1 v_5 u_5^3$ of $C_{0,1}$ is contracted to a single vertex and v_2, u_2^2 are degree-2 vertices of $B \backslash J_1$. Thus the claim is proved. ■

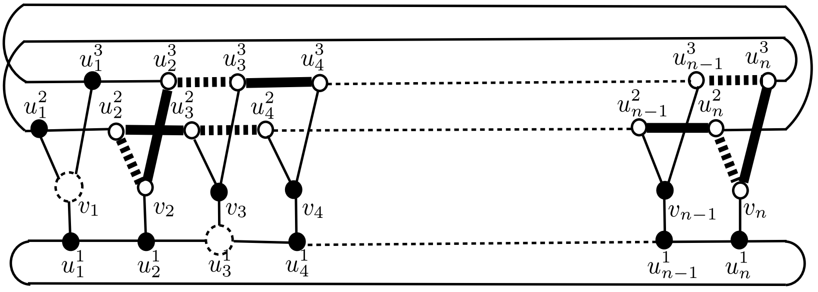

Fig. 14. Subcase 2.1: $n \geq 7$ and $j = 3$. (Legends: Dotted-circles v_1 and u_3^1 are two connector-vertices, hollow-circles are vertices of Q and solid-circles are vertices not in Q.)

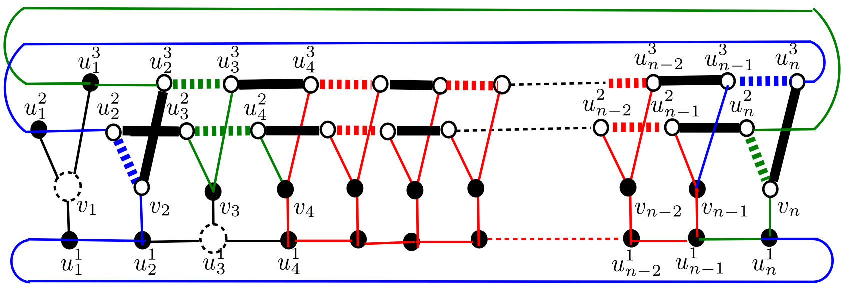

Fig. 15. Subcase 2.1: $B \backslash J_1$ with $n \geq 7$ and $j = 3$. (Legends: $E(X_0)$ is red, $E(X_1) - E(X_0)$ is green, $E(X_2) - (E(X_0) \cup E(X_1))$ is blue.) (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Since $X_0 \cup X_1$ is a spanning subgraph of B_0 with two components $C_{0,1} \cup C_{1,1}$ and $C_{0,2}$, by Claim 3.2.1, it is easy to see that $\overline{B \backslash J_1}/(X_0 \cup X_1) = B_1/X_1$ is a double star multipole. Thus, $\overline{B \backslash J_1}$ is $(4, DMS)$-girth-degenerate.

Case 2. Consider the flower snark F_n for $n \geq 7$.

Subcase 2.1. $j = 3$.

Let $Q = u_2^3 u_3^3 \cdots u_n^3 v_n u_n^2 u_{n-1}^2 \cdots u_2^2 v_2 u_2^3$, and J_1, J_2 be a pair of edge-disjoint perfect matchings of Q. (See Fig. 14, in which bold-lines are J_1-edges, and dashed-lines are J_2-edges.)

By the same reason as that π in Eq. (2) is an automorphism of the flower snark F_n with $\pi(J_1) = J_2$, one has that $\overline{B \backslash J_1} \cong \overline{B \backslash J_2}$. It is sufficient to consider $B \backslash J_1$ (see Fig. 15).

Let

$$C_{0,l} = \begin{cases} v_l u_l^1 u_{l+1}^1 v_{l+1} u_{l+1}^3 u_l^3 v_l, & \text{if } l \text{ is even and } l \in \{4, \ldots, n-1\}; \\ v_l u_l^1 u_{l+1}^1 v_{l+1} u_{l+1}^2 u_l^2 v_l, & \text{if } l \text{ is odd and } l \in \{5, \ldots, n-2\}, \end{cases}$$

$$C_{1,1} = u_1^3 u_n^2 v_n u_n^1 u_{n-1}^1 \cdots u_5^1 u_4^1 v_4 u_4^2 u_3^2 v_3 u_3^3 u_2^3 u_1^3, \quad C_{2,1} = v_{n-1} u_{n-1}^3 u_n^3 u_1^2 u_2^2 v_2 u_2^1 u_1^1 u_n^1 u_{n-1}^1 v_{n-1},$$

and

$$X_0 = \bigcup_{l=4}^{n-2} C_{0,l}, \quad X_1 = C_{1,1}, \quad X_2 = C_{2,1}.$$

Claim 3.2.2. *Each circuit $C_{0,\ell}$ of X_i corresponds to a 4^--circuit in B_i for $i = 0, 1, 2$.*

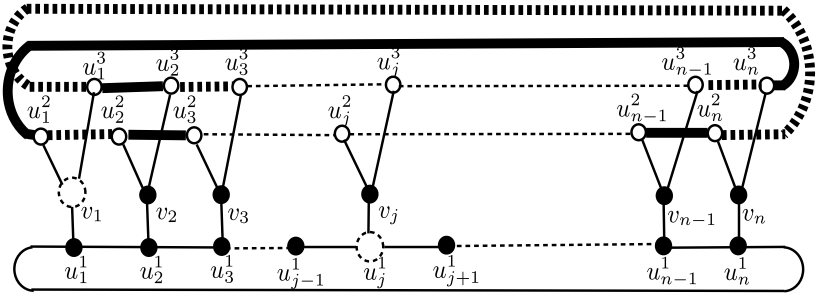

Fig. 16. Subcase 2.2: $n \geq 7$ and $4 \leq j \leq \frac{n+1}{2}$ (Legends: Dotted-circles v_1 and u_j^1 are two connector-vertices, hollow-circles are vertices of Q and solid-circles are vertices not in Q.)

Proof. Since $\{u_2^3, u_3^3, \ldots, u_n^3, v_n, u_n^2, u_{n-1}^2, \ldots, u_2^2, v_2\}$ $(= V(Q))$ is the set of degree-2 vertices in $B\backslash J_1$ (hollow-circles in the figure), each circuit of X_0 corresponds to a 4^--circuit in B_0.

Note that $C_{1,1}$ corresponds to a 4^--circuit in B_1 since the path $u_{n-1}^1 \cdots u_5^1 u_4^1 v_4$ in $C_{1,1}$ is contracted to a single vertex of B_1; $C_{2,1}$ corresponds to a 4^--circuit in B_2 as the path $u_n^1 u_{n-1}^1 v_{n-1}$ in $C_{2,1}$ is contracted to a single vertex in B_2. Thus the claim is proved. ■

Since $X_0 \cup X_1 \cup X_2$ is a spanning subgraph of B_0 with one components, by Claim 3.2.2, $\overline{B\backslash J_1}/(X_0 \cup X_1 \cup X_2) = B_2/X_2$ is a star multipole, thus $\overline{B\backslash J_1}$ is (4, MS)-girth-degenerate.

Subcase 2.2. $4 \leq j \leq \frac{n+1}{2}$.

Let

$$Q = u_1^2 u_2^2 \cdots u_n^2 u_1^3 u_2^3 \cdots u_n^3 u_1^2,$$

and J_1, J_2 be a pair of edge-disjoint perfect matchings of Q. (See Fig. 16, where bold-lines are J_1-edges and dashed-lines are J_2-edges.)

By the same reason as that π in Eq. (2) is an automorphism of the flower snark F_n with $\pi(J_1) = J_2$, one has that $\overline{B\backslash J_1} \cong \overline{B\backslash J_2}$. It is sufficient to consider $B\backslash J_1$ (see Fig. 17).

Let

$$C_{0,l} = \begin{cases} v_l u_l^1 u_{l+1}^1 v_{l+1} u_{l+1}^3 u_l^3 v_l, & \text{if } l \text{ is even and } l \in \{2, \ldots, j-2\} \cup \{j+1, \ldots, n-1\}; \\ v_l u_l^1 u_{l+1}^1 v_{l+1} u_{l+1}^2 u_l^2 v_l, & \text{if } l \text{ is odd and } l \in \{3, \ldots, j-2\} \cup \{j+1, \ldots, n-2\}, \end{cases}$$

$$C_{1,1} = \begin{cases} u_1^1 u_2^1 \cdots u_{j-1}^1 v_{j-1} u_{j-1}^3 u_j^3 v_j u_j^2 u_{j+1}^2 v_{j+1} u_{j+1}^1 u_{j+2}^1 \cdots u_n^1 u_1^1, & \text{if } j \text{ is odd}; \\ u_1^1 u_2^1 \cdots u_{j-1}^1 v_{j-1} u_{j-1}^2 u_j^2 v_j u_j^3 u_{j+1}^3 v_{j+1} u_{j+1}^1 u_{j+2}^1 \cdots u_n^1 u_1^1, & \text{if } j \text{ is even}, \end{cases}$$

and

$$X_0 = (\bigcup_{l=2}^{j-2} C_{0,l}) \cup (\bigcup_{l=j+1}^{n-1} C_{0,l}), \quad X_1 = C_{1,1}.$$

Claim 3.2.3. *The subgraph X_i corresponds to a union of 4^--circuits in B_i for $i = 0, 1$.*

Proof. Since each one in $\{u_1^2, u_2^2, \ldots, u_n^2, u_1^3, u_2^3, \ldots, u_n^3\}$ $(= V(Q))$ is a degree-2 vertex of $B\backslash J_1$ (hollow-circles in the figure). Hence, it is easy to see that X_0 corresponds to the union of 4^--circuit in $B_0 = \overline{B\backslash J_1}$.

Note that $C_{1,1}$ corresponds to a 4^--circuit in B_1 since each of the paths $u_2^1 u_3^1 \cdots u_{j-1}^1$ and $u_{j+1}^1 u_{j+2}^1 \cdots u_n^1$ in $C_{1,1}$ is contracted to a single vertex in B_1 and u_1^2, u_2^2, u_1^3 are degree-2 vertices. Thus the subgraph X_1 corresponds to a union of 4^--circuits in B_1. ■

Since $X_0 \cup X_1$ is a spanning subgraph of B_0 with one component, by Claim 3.2.3, B_1/X_1 is a star multipole, thus, $\overline{B\backslash J_1}$ is (4, MS)-girth-degenerate. □

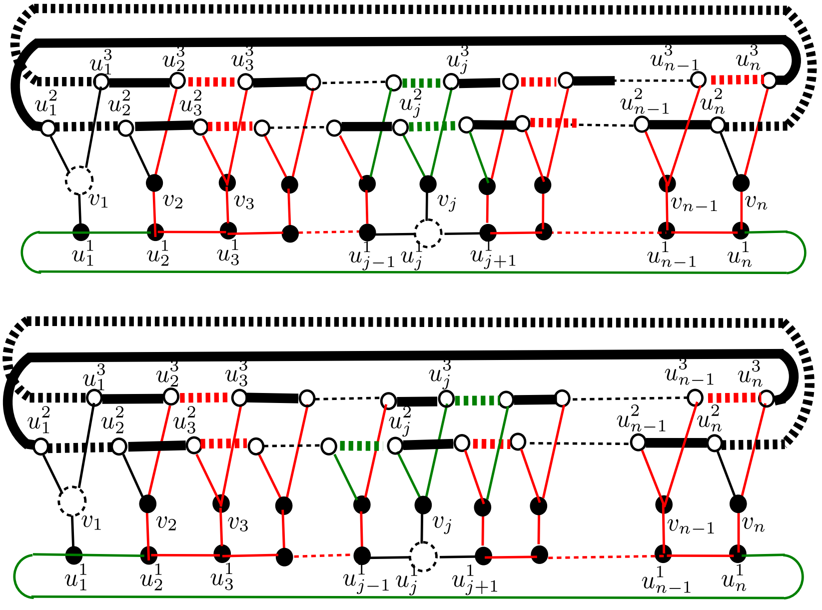

Fig. 17. Subcase 2.2: $B\backslash J_1$ with $n \geq 7$ and $4 \leq j \leq \frac{n+1}{2}$. The top one: j is odd, the bottom one: j is even. (Legends: $E(X_0)$ is red, $E(X_1) - E(X_0)$ is green.) (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4. Extensions and remarks

The method introduced in [26,27] can be further generalized to construct a large family of superposition snarks (see Definition 4.1 and Theorem 4.2). And, as an extension of Theorem 1.2, the B-F-Conjecture is also verified for some larger families of such snarks.

Definition 4.1. (1) Let G be a snark and R be an even subgraph of G, $\mathcal{V}$ be a set of supervertices and $\mathcal{E}$ be a set of proper superedges. The graph $K(G, R, \mathcal{V}, \mathcal{E})$ is a *superposition snark* if it is obtained from G by replacing vertices and edges of R with members of $\mathcal{V}$ and $\mathcal{E}$.

(2) The graph G is called the *frame* of $K(G, R, \mathcal{V}, \mathcal{E})$.

(3) And $R(\mathcal{V}, \mathcal{E})$ is the subgraph of $K(G, R, \mathcal{V}, \mathcal{E})$ induced by all superedges and supervertices around R. That is, $R(\mathcal{V}, \mathcal{E}) = K(G, R, \mathcal{V}, \mathcal{E}) - (G - V(R))$.

By Lemma 2.13, we have the following theorem.

Theorem 4.2. *Every graph $K(G, R, \mathcal{V}, \mathcal{E})$ defined in Definition 4.1-(1) is a snark.*

4.1. Berge–Fulkerson coloring for flower-expanded superposition snarks

The family $\mathcal{B}_K$ of Kochol flower superedges is generalized as follows.

Definition 4.3 (*Flower Superedges*)**.** Let F_n be a flower snark given in Definition 2.15, and u, v be two vertices with $d(u, v) \geq 3$. $(F_n)_{u,v}$ is the same as Definition 2.16.

(i) The set $\mathcal{B}_K$ (see Definition 2.16) of superedges can be further extended as follows.

$$\mathcal{B} = \{(F_n)_{u,v} : \ u, v \in V(F_n), d(u, v) \geq 3 \text{ for } n = 2k + 1 \text{ and } k \geq 2\}.$$

That is, connector-vertices u and v can be located in any subset V_{I}, V_{II}, V_{III} and V_{C}.

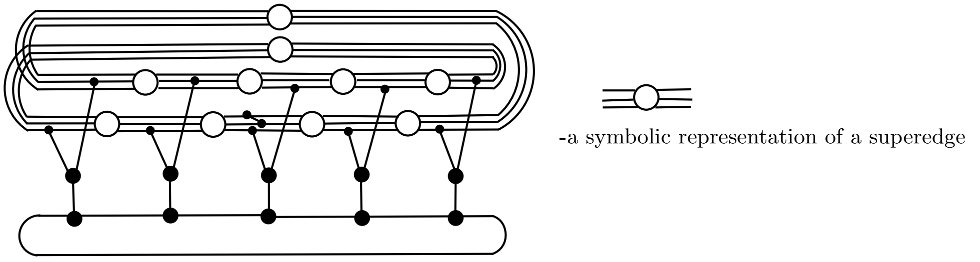

Fig. 18. A graph $K(G, R, \mathcal{V}, \mathcal{E})$ given in Definition 4.5.

(ii) Superedges constructed in (i) are called *flower superedges*.

It is obvious that $\mathcal{B}_K \subsetneq \mathcal{B}$.

With a similar proof of Lemma 3.1, we have the general lemma.

Lemma 4.4. *If $B \in \mathcal{B}$ (a flower-superedge in Definition 4.3), then*
(1) B contains an even subgraph $Q = J_1 \cup J_2$ with J_i being a matching,
(2) for each $i \in \{1, 2\}$, $\overline{B \backslash J_i}$ is either $(4, MS)$-girth-degenerate or $(4, DMS)$-girth-degenerate.

Definition 4.5. Let $K(G, R, \mathcal{V}, \mathcal{E})$ be a superposition snark with the frame G and the even subgraph R. Then $K(G, R, \mathcal{V}, \mathcal{E})$ (see Definition 4.1-(1)) is *flower-expanded* if $\mathcal{V} \subseteq \{A, A'\}$ and $\mathcal{E} = \mathcal{B}$ (defined in Definitions 2.14 and 4.3). That is,
(1) every edge of R is replaced with a flower-superedge $B \in \mathcal{B}$,
(2) and every vertex of R is replaced with A (Fig. 1) or A' (Fig. 2).

Fig. 18 is an example of $K(G, R, \mathcal{V}, \mathcal{E})$, where G is the flower snark F_5, R is a circuit of length 10 induced by the vertices in level II and level III, the set of supervertices $\mathcal{V}$ consists of A and A', and superedges are any member of $\mathcal{B}$.

With the same proof of Theorem 1.2, we have the following general lemma.

Lemma 4.6. *Let $K(G, R, \mathcal{V}, \mathcal{E})$ be a superposition snark defined in Definition 4.1-(1) with the frame G and the even subgraph R. If*
(1) the subgraph $R(\mathcal{V}, \mathcal{E})$ (defined in Definition 4.1-(3)) contains an even subgraph $Q = J_1 \cup J_2$ with J_i being a matching,
(2) for every $i \in \{1, 2\}$ and for every superedge B (contained in $R(\mathcal{V}, \mathcal{E})$), the suppressed multipole $\overline{B \backslash J_i}$ is either $(4, MS)$-girth-degenerate or $(4, DMS)$-girth-degenerate.
Then, for each $i \in \{1, 2\}$, $\overline{K(G, R, \mathcal{V}, \mathcal{E}) \backslash J_i}$ is $(4, G/R)$-girth-degenerate.

By Lemma 2.1 or Lemma 2.2, the combination of Lemmas 4.4 and 4.6 implies the following result, which generalizes Theorem 1.2.

Theorem 4.7. *Let $K(G, R, \mathcal{V}, \mathcal{E})$ be a superposition snark defined in Definition 4.1-(1) with the frame G and the even subgraph R. If*
(1) G/R admits nowhere-zero 4-flow and
(2) $K(G, R, \mathcal{V}, \mathcal{E})$ is flower-expanded (see Definition 4.5), then $K(G, R, \mathcal{V}, \mathcal{E})$ is Berge–Fulkerson colorable.

Proof. Similar to the final step of the proof of Theorem 1.2. $\square$

By Theorem 4.7, we may verify the B-F-conjecture for a flower-expanded superposition snark $K(G, R, \mathcal{V}, \mathcal{E})$ as long as G/R admits a nowhere-zero 4-flow (by applying Lemma 2.6). There are many graphs G satisfying such property. The following is a short list of flower-expanded superposition snark families (certainly, they contain the family of superposition snarks in our main theorem, and the most noticeable family of cyclically 6-edge-connected superposition snarks constructed by Kochol in [26,28]).

Corollary 4.8. *Let $K(G, R, \mathcal{V}, \mathcal{E})$ be a flower-expanded superposition snark with the frame G and the even subgraph R. Then $K(G, R, \mathcal{V}, \mathcal{E})$ admits a B-F-coloring if one of the followings holds.*

(1) G is the flower snark, and R is the circuit induced by vertices in level II and level III in Fig. 3.

(2) R has at most 2 components, $G - E(R)$ is acyclic.

(3) R has 3 components, $G - E(R)$ is acyclic, and, G/R is not the Petersen graph.

(4) G is a permutation graph and R is either the chordless 2-factor or one component of the 2-factor.

(5) G is any cyclically 4-edge-connected snark and R is any 2-factor of G.

(6) G is critical and R is any even subgraph of G. (A snark G is critical if G/e admits a nowhere zero 4-flow for any edge $e \in E(G)$.)

Proof. For (1) and (4), the contracted graph G/R is obviously 4-girth-degenerate. (2) and (3) are proved by Lemma 2.7. For (5), by Jaeger Theorem (Theorem 2.8), every 4-edge-connected graph admits a nowhere-zero 4-flow, and G/R is 4-edge-connected which implies G/R admits a nowhere zero 4-flow. For (6), the contracted graph G/R admits a nowhere-zero 4-flow obviously. □

Remark. The main theorem (Theorem 1.2) of the paper is a special case of (1) in Corollary 4.8 (Petersen graph is obtained from the flower snark F_3 by contracting the triangle). If R is a dominating circuit, then it is a special case of (2) in Corollary 4.8. It was conjectured by Thomassen that every cyclically 4-edge connected cubic graph contains a dominating circuit [39].

Acknowledgments

This work was supported by the National Science Foundation, USA (No. DMS-1700218) and the National Natural Science Foundation of China (Nos. 11971054, 11731002).

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