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Factorizations of Regular Graphs

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Let G be a k-regular graph of order $2n$ such that $k \geqslant n$. Hilton (*J. Graph Theory*, **9** (1985), 193–196) proved that G contains at least $\lfloor k/3 \rfloor$ edge-disjoint 1-factors. Hilton's theorem is improved in this paper that G contains at least $\lfloor k/2 \rfloor$ edge-disjoint 1-factors. The following result is also proved in this paper: Let G be a 2-connected, k-regular, non-bipartite graph of order at most $3k-3$ and x, y be a pair of distinct vertices. If $G\backslash\{x, y\}$ is connected, then G contains an (x, y)-Hamilton path. © 1992 Academic Press, Inc.

We use the notations of [BM]. Let $G=(V, E)$ be a graph with vertex set V and edge set E. A p-factor of a graph G is a p-regular spanning subgraph. Let G be a k-regular graph of order $2n$ and $\{p_1, ..., p_r\}$ a set of positive integers such that $p_1+\cdots+p_r=k$. If $H_1, ..., H_r$ are edge-disjoint regular spanning subgraphs of G with degree $p_1, ..., p_r$, respectively, then $\{H_1, ..., H_r\}$ is called a $(p_1, ..., p_r)$-factorization of G.

The following theorem was proved by Hilton:

THEOREM A ([H] or See [Z]). *Let G be a k-regular graph of order $2n$. (i) If $k \geqslant n$, then G contains at least $\lfloor n/3 \rfloor$ edge-disjoint 1-factors.* (ii) *Let $p_1, ..., p_s$ be odd positive integers and $p_{s+1}, ..., p_r$ be even positive integers such that $p_1+\cdots+p_r=k \geqslant n$ and $s \leqslant \lfloor n/3 \rfloor$; then G is $(p_1, ..., p_r)$-factorizable.*

In this paper, we prove the following theorem which improves the theorem of Hilton.

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THEOREM B (The Main Theorem). *Every k-regular graph of order $2n$ contains at least $\lfloor k/2 \rfloor$ edge-disjoint 1-factors if $k \geqslant n$.*

Let D be a subgraph of G and u a vertex of G. The set of vertices in D adjacent to u is denoted by $N_D(u)$. Let $P = v_1 \cdots v_p$ be a path of G. We denote

$$N_P^{+1}(u) = \{v_{i+1} \in V(P) : v_i \in N_P(u)\}$$

and

$$N_P^{-1}(u) = \{v_{i-1} \in V(P) : v_i \in N_P(u)\}.$$

Let H be a subgraph of G and X, Y a pair of disjoint vertex subsets of G. The set of edges of H joining X and Y is denoted by $E_H(X, Y)$ and the number of edges in the set $E_H(X, Y)$ is denoted by $e_H(X, Y)$. A graph G is called *Hamiltonian connected* if G contains an (x, y)-Hamilton path for every pair of vertices x and y of G.

The following results are basic lemmas in the proof of the main theorem.

LEMMA 1 (Tutte [T]). *If G is a graph containing no 1-factor, then G must have a vertex subset S such that the number of odd components of $G \backslash S$ is greater than the cardinality of S.*

LEMMA 2 (Wallis [W], or see [Pi]). *Let G be a d-regular graph of even order which contains no 1-factor. Let S be a vertex subset of order s such that the number r of odd components of $G \backslash S$ is greater than s, and r^+ the number of odd components of order at least $d+1$ of $G \backslash S$. Then*

1. $r \equiv s \bmod 2$;
2. $r \geqslant s + 2$;
3. $r^+ \geqslant 3$ *when* $s \geqslant 1$;
4. $|V(G)| \geqslant s + r + dr^+$.

LEMMA 3 (Dirac [D]). *If G is a graph of order at most 2δ and δ is the minimum degree of G, then G contains a Hamilton cycle.*

LEMMA 4 (Lovász [LL 10.24]). *If G is a graph of order at most $2\delta - 1$ and δ is the minimum degree of G, then G is Hamiltonian connected.*

LEMMA 5 (Jung [J]). *Every 3-connected, k-regular, non-bipartite graph of order at most $3k-1$ is Hamiltonian connected.*

LEMMA 6 *Let G be a 2-connected graph with a 2-vertex-cut and minimum degree δ. Let x, y be a pair of distinct vertices such that $G\backslash\{x, y\}$ is connected. If G is of order at most $3\delta-3$, then G contains an (x, y)-Hamilton path.*

Proof. Let $\{u, v\}$ be a 2-vertex-cut of G. Since the minimum degree of G is δ and G contains at most $3\delta-3$ vertices, $G\backslash\{u, v\}$ has only two components. For each 2-cut $\{u, v\}$ of G, let C^1_{uv} and C^2_{uv} be the compontents of $G\backslash\{u, v\}$, and H^1_{uv} the subgraph of G induced by $C^i_{uv} \cup \{u, v\}$ (for $i=1, 2$). Since each component of $G\backslash\{u, v\}$ contains at least $\delta-1$ vertices and $|V(G)| \leqslant 3\delta-3$, we have that $\delta \geqslant 3$. If $\delta=3$, then G is a graph H or $H+uv$ (see Fig. 1). It is easy to see that the lemma is true in this case. Thus we assume that

$$\delta \geqslant 4$$

and therefore each component of $G\backslash\{u, v\}$ contains at least 3 vertices. It is also evident that the lemma is true if both subgraphs H^1_{uv} and H^2_{uv} are complete for some 2-cut $\{u, v\}$ of G. Let G be a 2-connected graph and $\{x, y\}$ a pair of vertices of G such that the following hold:

(1) the minimum degree of G is δ and $|V(G)| \leqslant 3\delta-3$,

(2) G has a 2-cut,

(3) $G\backslash\{x, y\}$ is connected,

(4) subject to (1), (2), and (3), G has no (x, y)-Hamilton path,

(5) subject to (1), (2), (3), and (4), $|E(G)|$ is as large as possible.

I. For each 2-cut $\{u, v\}$ of G, we claim that $C_1=C^1_{uv}$ and $C_2=C^2_{uv}$ are cliques. Assume that there are a pair of non-adjacent vertices w' and w'' in C_1. By the choice of the graph G, the graph $G+w'w''$ contains an (x, y)-Hamilton path P, where the edge $w'w''$ must an edge somewhere in P.

Let $H_i=H^i_{uv}$ $(i=1, 2)$. It is easy to see that

$$\delta+1 \leqslant |V(H_i)| \leqslant 2\delta-2$$

for $i=1, 2$ because all neighbors of each vertex of C_i are contained in H_i.

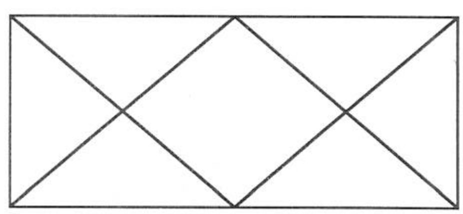

FIGURE 1

Since G does not contain an (x, y)-Hamilton path,

$$N_Q^{+1}(w') \cap N_Q(w'') = \varnothing \text{ (see Fig. 2)}$$

and

$$d_Q(w') + d_Q(w'') \leqslant |V(Q)| + 1$$

for any segment Q of $p \backslash \{w', w''\}$. Since w' and w'' belong to the same component $C_\mu (\mu = 1$ or $2)$, $N(w')$ and $N(w'') \subseteq V(H_\mu)$. For any $\{i, j\} = \{1, 2\}$, $P \backslash [\{w', w''\} \cup V(C_i)]$ consists of at most three segments in H_j. Thus

$$d_{H_\mu}(w') + d_{H_\mu}(w'') \leqslant |V(H_\mu) \backslash \{w', w''\}| + 3 \leqslant 2\delta - 1.$$

This contradicts the fact that $d_{H_\mu}(w') + d_{H_\mu}(w'') \geqslant 2\delta$.

II. By I, C_{uv}^1 and C_{uv}^2 are cliques for each 2-cut $\{u, v\}$ of G. We consider the following two representative cases.

Case 1. $\{x, y\} \subseteq H_{uv}^1$ for some 2-cut $\{u, v\}$ of G.

It is evident that H_{uv}^2 contains a (u, v)-Hamilton path P_0 since C_{uv}^2 is a clique and G is 2-connected. Since G is 2-connected again, there are a pair of disjoint paths P_1 and P_2 joining $\{x, y\}$ and $\{u, v\}$ in G. Obviously both P_1 and P_2 are contained in H_{uv}^1. Choose P_1 and P_2 such that $|V(P_1)| + |V(P_2)|$ is as large as possible. If $V(H_{uv}^1) \backslash (P_1 \cup P_2) = \varnothing$, then $P_0 \cup P_1 \cup P_2$ is an (x, y)-Hamilton path of G. This contradicts the assumption. Thus, we assume that $V(H_{uv}^1) \backslash (P_1 \cup P_2) \neq \varnothing$. Since C_{uv}^1 is a clique, $|V(P_1) \cap V(C_{uv}^1)| \leqslant 1$ and $|V(P_2) \cap V(C_{uv}^1)| \leqslant 1$. This implies that $\{x, y\}$ is a 2-cut of G, a contradiction.

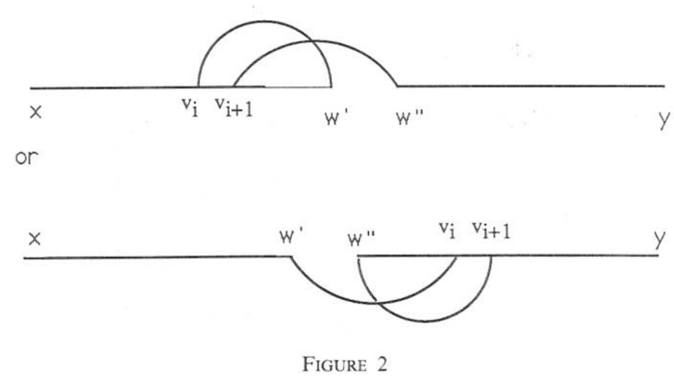

FIGURE 2

Since x and y cannot be in the same subgraph $H^i_{uv} = G(C^i_{uv} \cup \{u, v\})$ for each $i = 1, 2$ and each 2-cut $\{u, v\}$, we have that

(i) neither x nor y belongs to any 2-cut of G,

(ii) x and y belong to different components of $G \backslash \{u, v\}$ for any 2-cut $\{u, v\}$.

Hence, the following case is the only remaing case.

Case 2 $z_1 \in C^1_{uv}$ and $z_2 \in C^2_{uv}$ for $\{z_1, z_2\} = \{x, y\}$.

By (i), $G \backslash \{z_i\}$ is 2-connected and there are two distinct vertices $\{a_{iu}, a_{iv}\}$ in $C^i_{uv} \backslash \{z_i\}$ such that $a_{iw} \in V(C^i_{uv}) \cap N(w)$ for each $w \in \{u, v\}$ and each $i = 1, 2$. Since each C^i_{uv} $(i = 1, 2)$ is a clique, for each $w \in \{u, v\}$ and each C^i_{uv} $(i = 1, 2)$, there is a (z_i, w)-Hamilton path $Q^i_w = z_i \cdots a_{iw} w$ in the subgraph of G induced by $C^i_{uv} \cup \{w\}$. Furthermore, uv is not an edge of G for otherwise $Q^1_u \cup Q^2_v \cup \{uv\}$ is an (x, y)-Hamilton path of G.

If $\{a_{2u}, a_{2v}\}$ is a 2-cut of G, then $\{u, v\}$ belong to a component of $G \backslash \{a_{2u}, a_{2v}\}$ which, by I, it is clique. This contradicts that uv is not an edge of G. Thus, $G \backslash \{a_{2u}, a_{2v}\}$ is connected. Since $|V(C^2_{uv})| \geqslant 3$, there is a vertex b in $C^2_{uv} \backslash \{a_{2u}, a_{2v}\}$ adjacent to either u or v. Without loss of generality, let $b \in N(u)$. Since C^2_{uv} is a clique, let $Q_1 = z_2 \cdots bu$ and $Q_2 = va_{2v} \cdots a_{2u}u$ be two paths in H^2_{uv} such that $V(Q_1) \cap V(Q_2) = \{u\}$ and $V(Q_1) \cup V(Q_2) = V(H^2_{uv})$. Then $Q_1 \cup Q_2 \cup Q^1_v$ is a (z_1, z_2)-Hamilton path in G. This contradicts the assumption and completes the proof. ∎

By applying Lemma 5 and Lemma 6, we have the following theorem which was originally proved in [ZZ].

THEOREM C (Zhang and Zhu [ZZ]). *Let G be a 2-connected, k-regular, non-bipartite graph of order at most $3k - 3$ and x, y be a pair of distinct vertices. If $G \backslash \{x, y\}$ is connected, then G contains an (x, y)-Hamilton path.*

LEMMA 7. *Let G be a graph of order at most $2\delta - 4$ and δ be the minimum degree of G.*

(i) *If u, v, w, x are four distinct vertices of G, then there are two disjoint paths P_1 and P_2 joining u and v, w and x, respectively, in G and the union of P_1 and P_2 spans G.*

(ii) *If u, v, w are three distinct vertices of G, then there is a Hamilton path in $G \backslash \{w\}$ joining u and v.*

Proof. (i) If $uv \in E(G)$, then let $G' = G \backslash \{u, v\}$. If $uv \notin E(G)$, then there is a vertex $z \in [N(u) \cap N(v)] \backslash \{w, x\}$ because

$$|N(u) \backslash \{w, x\}| + |N(v) \backslash \{w, x\}| \geqslant 2\delta - 4 > |V(G) \backslash \{u, v, w, x\}|.$$

Let $G'' = G \backslash \{u, v, z\}$. By Lemma 4, both G' and G'' are Hamiltonian connected and there exists a Hamilton path P_2 joining w and x in G' or G''. The path P_1 joining u and v is uv if $uv \in E(G)$ or uzv if $uv \notin E(G)$.

(ii) By Lemma 4, it is easy to see that $G^* = G \backslash \{w\}$ is Hamiltonian connected. ▮

LEMMA 8. *Let G be a 2-connected d-regular grah of order at most* $3d-4$ *and* V' *a vertex subset of G of order* 3. *If G is not a bipartite graph, then there is a Hamilton path of G joining two vertices of* V'.

Proof. By Theorem C, it is sufficient to show that there must be two vertices x and y of V' such that $G \backslash \{x, y\}$ is connected.

Let $V' = \{v_1, v_2, v_3\}$. Assume that $G \backslash \{v_i, v_j\}$ is disconnected for any pair of $i, j \in \{1, 2, 3\}$. Let C_1, C_2 be two disconnected parts of $G \backslash \{v_1, v_2\}$ and D_1, D_2 be two disconnected parts of $G \backslash \{v_1, v_3\}$. Without loss of generality, let $v_3 \in C_2$ and $v_2 \in D_1$. Then $G \backslash V'$ has three disconnected parts C_1, D_2, and $C_2 \cap D_1$. Obviously,

$$N(u) \subseteq [C_1 \cup \{v_1, v_2\}] \backslash \{u\} \qquad \text{for} \quad u \in V(C_1),$$
$$N(u) \subseteq [D_2 \cup \{v_1, v_3\}] \backslash \{u\} \qquad \text{for} \quad u \in V(D_2),$$

and

$$N(u) \subseteq [(C_2 \cap D_1) \cup \{v_1, v_2, v_3\}] \backslash \{u\} \qquad \text{for} \quad u \in V(C_2 \cap D_1).$$

Then $|V(C_1)| \geqslant d-1$, $|V(D_2)| \geqslant d-1$ and $|V(C_2 \cap D_1)| \geqslant d-2$. That is,

$$|V(G)| = |V(C_1)| + |V(D_2)| + |V(C_2 \cap D_1)| + |\{v_1, v_2, v_3\}| \geqslant 3d-1.$$

This contradicts that $|V(G)| \leqslant 3d-4$. ▮

LEMMA 9 (Peterson [P]). *Every 2k-regular graph contains k edge-disjoint 2-factors.*

The Proof of The Main Theorem. Let $\{F_1, ..., F_t\}$ be a maximum set of disjoint 1-factors in G. Let $h = k - t$ and $H = G \backslash E(F_1 \cup \cdots \cup F_t)$ which is an h-regular graph. The proof of this theorem is by contradiction. Suppose that $t < \lfloor k/2 \rfloor$. Thus H is of order at most $4h-4$.

An *even 2-factor* is a 2-factor such that each component of it is a cycle of even length. Obviously, any even 2-factor is a union of two disjoint 1-factors. We claim that the following statement (*) holds for any $F_\mu \in \{F_1, ..., F_t\}$:

$$H \cup F_\mu \text{ contains no even 2-factor.} \qquad (*)$$

Assume that $H \cup F_\mu$ contains an even 2-factor which is the union of two disjoint 1-factors F' and F''. We can replace F_μ of $\{F_1, ..., F_t\}$ by F', F'' and obtain a bigger set of disjoint 1-factors in G. This contradicts the choice of $\{F_1, ..., F_t\}$.

By Lemma 1, let S be a smallest vertex subset of order s such that the number of odd components if $H \backslash S$ is greater than s. Let $C_1, ..., C_r$ be the odd components of $H \backslash S$. Here $r > s$. If C is a component of $H \backslash S$ and v is a vertex of C, then $N(v) \subseteq [V(C) \cup S] \backslash \{v\}$. By the h-regularity of H, $|V(C) \cup S| \geqslant h + 1$ and hence

$$|V(C)V \geqslant h - s + 1$$

for any component C of $H \backslash S$. By Lemma 2, we must have that

$$\begin{aligned} 4h - 4 \geqslant |V(H)| \geqslant s + \left| \bigcup_{i=1}^{r} V(C_i) \right| \geqslant s + (h + 1 - s)r \\ \geqslant s + (h + 1 - s)(s + 2). \end{aligned}$$

That is $(s-2)(s-h+2) \geqslant 2$. Therefore either $s \leqslant 1$ or $s \geqslant h - 1$.

If $s \geqslant h - 1$, then

$$|V(H)| \geqslant s + r + hr^+$$

(by (4) of Lemma 2)

$$\geqslant (h-1) + (s+2) + 3h$$

(by (2) and (3) of Lemma 2)

$$\begin{aligned} &\geqslant (h-1) + ((h-1)+2) + 3h \\ &= 5h. \end{aligned}$$

This contradicts that $|V(H)| \leqslant 4h - 4$. So S must be either a single vertex or an empty set.

Case One. $s = 1$. *Let* $S = \{w\}$. If H is disconnected, then each component of H is of even order because of the choice of S. So $H \backslash S$ has at least four components. Since each component of $H \backslash S$ is of order at least h, H contains at least $4h + 1$ vertices and this contradicts that $|V(H)| \leqslant 4h - 4$. Therefore, H must be connected in this case. Moreover, $H(C \cup S)$ is not a clique for any component C of $H \backslash S$ and hence $|V(C)| \geqslant h + 1$. Thus $H \backslash S$ has exactly three components, C_1, C_2, and C_3, each of which is of odd order and for any $i = 1, 2, 3$,

$$|V(C_i)| \leqslant |V(H)| - |S| - |V(C_j)| - |V(C_{j'})|$$

(where $j, j' \neq i$)

$$\leqslant |V(H)| - 1 - 2(h+1)$$
$$\leqslant |V(H)| - 2h - 3.$$

Since $|V(H)|/2 \leqslant 2h-2$, we have that

$$|V(C_i| \leqslant \frac{|V(H)|}{2} - 5 \leqslant 2h - 7 \tag{1}$$

for any $i = 1, 2, 3$.

Since $|C_1|$ is odd, $e_{F_\mu}(C_1, V\backslash V(C_1))$ is odd for each $F_\mu \in \{F_1, ..., F_t\}$. We claim that there is $F_\mu \in \{F_1, ..., F_t\}$ such that $e_{F_\mu}(C_1, V\backslash V(C_1)) \geqslant 3$. If not, then $e_{F_\mu}(C_1, V\backslash V(C_1)) = 1$ for any $F_\mu \in \{F_1, ..., F_t\}$ and $\sum_\mu e_{F_\mu}(C_1, V\backslash(C_1)) = t < h+1 \leqslant |V(C_1)|$. So there must be a vertex v of C_1 such that the neighbor of v in each F_μ is contained in C_1, that is all vertices adjacent to v in G are contained in $V(C_1) \cup \{w\}$. But this implies that

$$|V(C_1)| \geqslant k \geqslant \frac{|V(H)|}{2}.$$

This contradicts that $|V(C_1)| \leqslant |V(H)|/2 - 5$ and therefore, our claim holds. Without loss of generality, let $e_{F_1}(C_1, V\backslash V(C_1)) \geqslant 3$.

Assume that $e_{F_1}(C_1, C_j) \neq 0$ for each $j = 2, 3$. Let $x_{21} \in N(w) \cap V(C_2)$, $x_{31} \in N(w) \cap V(C_2)$, and $x_{11}x_{22}$, $x_{12}x_{32}$ be edges of F_1 where $x_{ij} \in V(C_i)$ for $i = 1, 2$. By Lemmas 3 and 4, for $i = 1, 2, 3$, let P_i and Q_i be a pair of disjoint path and cycle in $H(C_i)$ such that either P_i is an (x_{i1}, x_{i2})-Hamilton path of $H(C_i)$ and Q_i is empty if $x_{i1} \neq x_{i2}$, or P_i is a single vertex x_{i1} and Q_i is a Hamilton cycle in $H(C_i)\backslash\{x_{i1}\}$ if $x_{i1} = x_{i2}$. Thus we obtain an even 2-factor

$$\{P_1 \cup P_2 \cup P_3 \cup \{wx_{21}, wx_{31}, x_{22}x_{11}, x_{12}x_{32}\}, Q_2, Q_3\}$$

in $H \cup F_1$ which contradicts the statement (*). So either $e_{F_1}(C_1, C_2) = 0$ or $e_{F_1}(C_1, C_3) = 0$. See Fig. 3.

Let $e_{F_1}(C_1, C_2) = 0$. Then $e_{F_1}(C_1, C_3) \geqslant 2$. If $e_{F_1}(C_2, C_3) \neq 0$, then the proof is the same as the case of $e_{F_1}(C_1, C_2) \neq 0$ and $e_{F_1}(C_1, C_3) \neq 0$ by exchanging C_1 and C_3. So we assume that $e_{F_1}(C_3, C_2) = 0$. Let wy_{21} be and edge of H joining w and C_2. Since C_2 is of odd order, $e_{F_1}(C_2, V\backslash V(C_2)) \neq 0$ and hence $e_{F_1}(C_2, w) \neq 0$. Let wy_{22} be an edge of F_1 joining w and C_2, and $y_{11}y_{31}$ and $y_{12}y_{32}$ pair of distinct edges of F_1 joining C_1 and C_3 (where

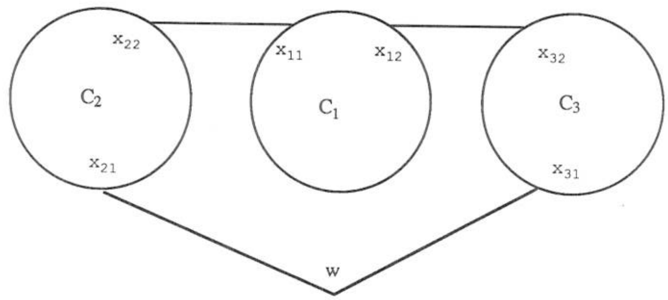

FIGURE 3

$y_{ij} \in V(C_i)$ for $j = 1, 2$). By Lemma 4, let R_i be a (y_{i1}, y_{i2})-Hamilton path in $H(C_i)$ for $i = 1, 2, 3$. Then we obtain an even 2-factor

$$\{R_1 \cup R_3 \cup \{y_{11}y_{31}, y_{12}y_{32}\}, R_2 \cup \{wy_{21}, wy_{22}\}\}$$

in $H \cup F_1$ and this contradicts the statement (*). See Fig. 4.

Case Two. $s = 0$. Since each component is of order at least $h+1$ and $|V(H)| \leqslant 4h-4$, H has at most three components. By Lemma 2, two components must be of odd order. Thus

$$h+1 \leqslant |V(C)| \leqslant 3h-5 \tag{2}$$

for any component C of H. The degree h of H must be an even integer because H has some odd components.

If C is an odd component of order at most $|V(H)|/2$ of H, we claim that there is $F_\mu \in \{F_1, ..., F_t\}$ such that $e_{F_\mu}(C, V \backslash V(C)) \geqslant 3$. We have that $e_{F_\mu}(C, V \backslash V(C))$ is odd since $|V(C)|$ is odd. Suppose that $e_{F_\mu}(C, V \backslash V(C)) = 1$ for every $F_\mu \in \{F_1, ..., F_t\}$. Then

$$|V(C)| \geqslant h+1 > t = \sum_\mu e_{F_\mu}(C, V \backslash V(C))$$

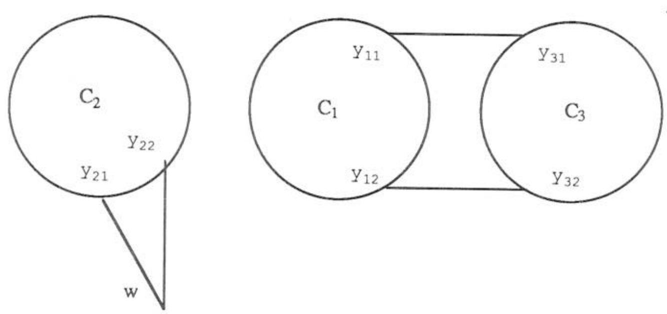

FIGURE 4

and there is a vertex v of C such that the neighbor of v in each F_μ is contained in C. Hence, all vertices adjacent to v in G are contained in C and

$$|V(C)| \geqslant k+1 > \frac{|V(H)|}{2}$$

which contradicts the assumption that $|V(C)| \leqslant |V(H)|/2$.

Subcase 1. H has two components and both components are blocks.

Let C_1 and C_2 be two odd components of H. Without loss of generality, let $|V(C_1)| \leqslant |V(H)|/2$ and F_1 a 1-factor such that $e_{F_1}(C_1, V \backslash V(C_1)) = e_{F_1}(C_1, C_2) \geqslant 3$ and $u_\mu v_\mu \in E_{F_1}(C_1, C_2)$ for $\mu = 1, 2, 3$. Note that

$$h+1 \leqslant |V(C_1)| \leqslant \frac{|V(H)|}{2} \leqslant 2h-2$$

and

$$\frac{|V(H)|}{2} \leqslant |V(C_2)| \leqslant 3h-5.$$

Since $H(C_2)$ is regular and of odd order, $H(C_2)$ cannot be a bipartite graph. By Lemma 8, there is a Hamilton path P_2 joining two vertices of $\{v_1, v_2, v_3\}$ in $H(C_2)$ and without loss of generality, let P_2 join v_1 and v_2. By Lemma 4, let P_1 be a (u_1, u_2)-Hamilton path in $H(C_1)$. Then $H \cup F_1$ contains a Hamilton cycle $P_1 \cup P_2 \cup \{u_1 v_1, u_2 v_2\}$ which is an even 2-factor and contradicts the statement (*).

Subcase 2. H has two components and one component is not a block.

Let C_1 and C_2 be two components of H. Without loss of generality, let C_2 be a non-block component and w a cut vertex of C_2. $C_2 \backslash \{w\}$ can have only two components because $|V(C_2)| \leqslant 3h-5$. Let D_1 and D_2 be the two components of $C_2 \backslash \{w\}$. Since H is h-regular and $H(D_i \cup w)$ is not a clique for $i = 1, 2$, $|V(C_1)|$, $|V(D_1)|$, and $|V(D_2)| \geqslant h+1$. Hence, for any $\{A, A', A''\} = \{C_1, D_1, D_2\}$,

$$\begin{aligned} |A| &\leqslant |V(H)| - |\{w\}| - |A'| - |A''| \\ &\leqslant |V(H)| - 1 - 2h - 2 \end{aligned}$$

(since $|V(H)|/2 \leqslant 2h-2$)

$$\begin{aligned} &\leqslant |V(H)| - 5 - \frac{|V(H)|}{2} \\ &= \frac{|V(H)|}{2} - 5 \\ &\leqslant 2h - 7. \end{aligned}$$

Since the degree h of H is an even number and the number of odd degree vertices in the subgraph $H(D_i \cup w)$ is even, $e_H(D_i, w)$ is even for $i=1, 2$. Let x_{i1}, x_{i2} be two vertices of D_i adjacent to w in H for $i=1, 2$. Since $|V(C_1)| \leqslant |V(H)|/2$, let F_1 be a 1-factor such that $e_{F_1}(C_1, V \backslash V(C_1)) \geqslant 3$.

If $e_{F_1}(C_1, D_1) \neq 0$ for both $i=1$ and 2, then let $y_1 z_1$ and $y_2 z_2$ be two edges of F_1 joining C_1 and D_1 and D_2 where $y_1, y_2 \in C_1$ and $z_i \in D_1$ for $i=1, 2$. Without loss of generality, assume that $x_{i1} \neq z_i$ for $i=1, 2$. By Lemma 4, let P_i be an (x_{i1}, z_i)-Hamilton path in $H(D_i)$ for $i=1, 2$ and P_0 be a (y_1, y_2)-Hamilton path in $H(C_1)$. Then $H \cup F_1$ contains a Hamilton cycle $P_0 \cup P_1 \cup P_2 \cup \{x_{11} w, x_{21} w, y_1 z_1, y_2 z_2\}$. This contradicts that $H \cup F_1$ contains no even 2-factor. See Fig. 5.

So we assume that $e_{F_1}(C_1, D_1) \geqslant 2$ and $e_{F_1}(C_1, D_2) = 0$. Let $E_{F_1}(C_1, D_1) = \{u_\mu, v_\mu): \mu = 1, 2, \ldots \ldots\}$.

(i) If $|V(D_2)|$ is odd, then by Lemma 4, let Q_0 be a (u_1, u_2)-Hamilton path in $H(C_1)$, Q_1 a (v_1, v_2)-Hamilton path in $H(D_1)$, and Q_2 an (x_{21}, x_{22})-Hamilton path in $H(D_2)$. Thus

$$\{Q_0 \cup Q_1 \cup \{u_1 v_1, u_2 v_2\}, Q_2 \cup \{x_{21} w, x_{22} w\}\}$$

is an even 2-factor in $H \cup F_1$ and this contradicts the statement (*). See Fig. 6.

(ii) If $|V(D_2)|$ is even and $e_{F_1}(C_1, w) \neq 0$ then let $u_0 w \in F_1$. Without loss of generality, assume that $v_1 \neq x_{11}$. By Lemma 4, let R_0 and R_1 be (u_0, u_1)- and (v_1, x_{11})-Hamilton paths in $H(C_1)$ and $H(D_1)$, respectively; and let R_2 be a Hamilton cycle in $H(D_2)$. Then $\{R_0 \cup R_1 \cup \{u_0 w, w x_{11}, u_1 v_1\}, R_2\}$ is an even 2-factor in $H \cup F_1$ and this contradicts (*) again. See Fig. 7.

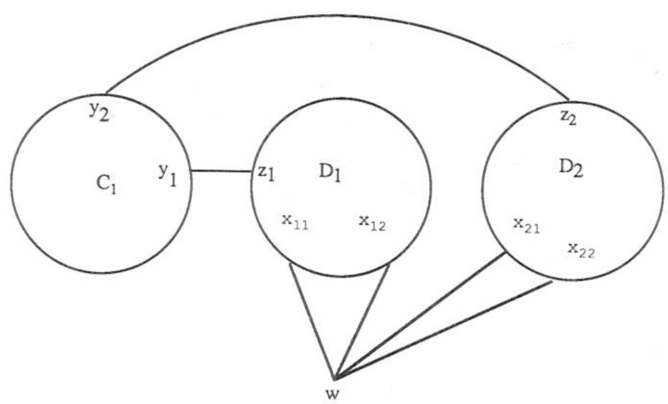

FIGURE 5

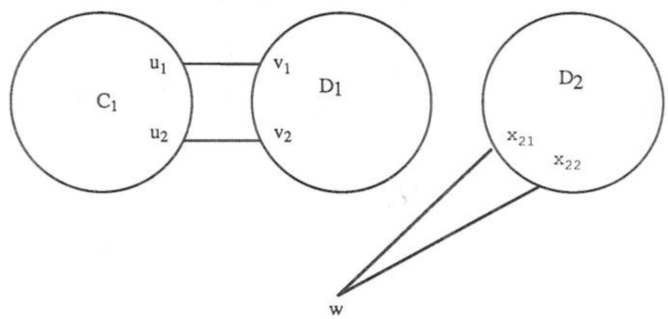

FIGURE 6

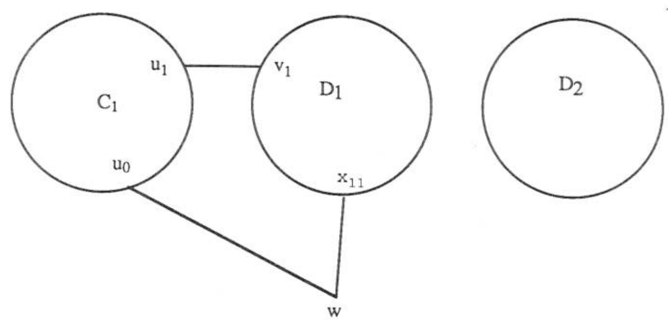

FIGURE 7

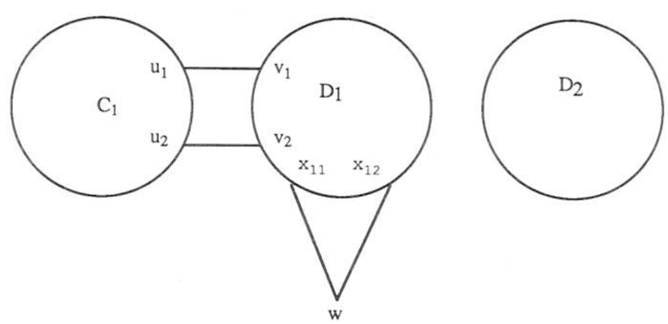

FIGURE 8

(iii) If $|V(D_2)|$ is even and $e_{F_1}(C_1, w)=0$, then $e_{F_1}(C_1, D_1)= e_{F_1}(C_1, V\backslash V(C_1))\geqslant 3$. When $\{v_1, v_2, v_3\}\cap\{x_{11}, x_{12}\}\neq\varnothing$, let $v_1=x_{11}$ be a vertex in this intersection and $v_2\in\{v_1, v_2, v_3\}\backslash\{x_{11}, x_{12}\}$. By (ii) of Lemma 7, let $S'=v_1$ and let S'' be a (v_2, x_{12})-Hamilton path in $H(D_1)\backslash\{v_1\}$. When $\{v_1, v_2, v_3\}\cap\{x_{11}, x_{12}\}=\varnothing$, by (i) of Lemma 7, let S' and S'' be a pair of disjoint paths joining v_1 and x_{11}, v_2 and x_{12}, respectively, in $H(D_1)$. By Lemma 3 and Lemma 4, let S_0 be a (u, u_2)-Hamilton path in $H(C_1)$ and S_2 be a Hamilton cycle in $H(D_2)$. Then $H\cup F_1$ contains an even 2-factor $\{S_0\cup S'\cup S''\cup\{u_1v_1, u_2v_2, wx_{11}, wx_{12}\}, S_2\}$ and this contradicts (*). See Fig. 8.

Subcase 3. H has three components, C_1, C_2, and C_3.

Let C_1 and C_2 be the odd components and C_3 the even component of H. Obviously,

$$\begin{aligned} h+1 &\leqslant |V(C_i)| \\ &\leqslant |V(H)| - V(C_j)| - |V(C_{j'})| \qquad \text{for } j, j'\neq 1 \\ &\leqslant |V(H)| - 2(h+1) \\ &\leqslant |V(H)| - \frac{|V(H)|}{2} - 4 \\ &= \frac{|V(H)|}{2} - 4 \\ &\leqslant 2h-6 \end{aligned}$$

for any $i=1, 2, 3$. We claim that there is an $F_\mu\in\{F_1, ..., F_t\}$ such that $e_{F_\mu}(C_1, V\backslash V(C_1))\geqslant 3$ and $e_{F_\mu}(C_2, V\backslash V(C_2))\geqslant 3$. If not, we have that either $e_{F_\mu}(C_1, V\backslash V(C_1))=1$ or $e_{F_\mu}(C_2, V\backslash V(C_2))=1$ for any $F_\mu\in\{F_1, ..., F_t\}$ because C_1 and C_2 are odd components and $e_{F_\mu}(C_i, V\backslash V(C_i))$ is odd for $i=1, 2$. Let

$$I_1=\{\mu: e_{F_\mu}(C_1, V\backslash V(C_1))=1\}$$

and

$$I_2=\{\mu: e_{F_\mu}(C_2, V\backslash V(C_2))=1\}.$$

Here $\{1, ..., t\}=I_1\cup I_2$ and $t\leqslant |I_1|+|I_2|$. The graph $H_i=H\cup(\bigcup_{\mu\in I_i} F_\mu)$ is $(h+|I_i|)$-regular for $i=1, 2$. Since $e_{F_\mu}(C_1, V\backslash V(C_1))=1$ for any $\mu\in I_1$ and

$$|V(C_1)|\geqslant h+1>t\geqslant |I_1|=\sum_{\mu\in I_i} e_{F_\mu}(C_1, V\backslash V(C_1)),$$

there must exist a vertex v of C_1 such that $e_{F_\mu}(v, V\backslash V(C_1))=0$ for each $\mu \in I_1$. Hence all vertices adjacent to v in H_1 are contained in C_1 and $|V(C_1)| \geqslant h+|I_1|+1$. Similarly $|V(C_2)| \geqslant h+|I_2|+1$. But

$$\begin{aligned}|V(C_3| &= |V(G)| - |V(C_1)| - |V(C_2)| \\ &\leqslant 2k-(h+|I_1|+1)-(h+|I_2|+1) \\ &= 2k-2h-|I_1|-|I_2|-2 \\ &\leqslant 2k-2h-t-2 \\ &= k-h-2\end{aligned}$$

(as $h+t=k$)

$$<h-2$$

(as $k<2h$). This contradicts that $|V(C_3)| \geqslant h+1$ and our claim holds.

Without loss of generality, let F_1 be a 1-factor such that $e_{F_1}(C_1, V\backslash V(C_1)) \geqslant 3$ and $e_{F_1}(C_2, V\backslash V(C_2)) \geqslant 3$. If $e_{F_1}(C_1, C_2) \geqslant 2$, then let edges $x_{11}x_{21}, x_{12}x_{22} \in E_{F_1}(C_1, C_2)$. By Lemmas 3 and 4, let P_i be an (x_{i1}, x_{i2})-Hamilton path in C_i for $i=1, 2$, and let P_3 be a Hamilton cycle in C_3. Thus $\{P_1 \cup P_2 \cup \{x_{11}x_{21}, x_{12}x_{22}\}, P_3\}$ is an even 2-factor in $H \cup F_1$ and this contradicts the statement (*). See Fig. 9.

So we have that $e_{F_1}(C_1, C_2) \leqslant 1$ and hence $e_{F_1}(C_1, C_3) \geqslant 2$ and $e_{F_1}(C_2, C_3) \geqslant 2$. Let edges $z_{11}x_{31}, z_{12}x_{32} \in E_{F_1}(C_1, C_2)$ and $z_{21}y_{31}, z_{22}y_{32} \in E_{F_1}(C_2, C_3)$. By Lemma 4, et Q_i be a (z_{i1}, z_{i2})-Hamilton path in C_i for $i=1, 2$. By (i) of Lemma 7 let Q_3, Q_4 be a pair of disjoint (x_{31}, y_{31})- and (x_{32}, y_{32})-paths of C_3. Thus the Hamilton cycle $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup \{z_{11}x_{31}, z_{12}x_{32}, z_{21}y_{31}, z_{22}y_{32}\}$ is an even 2-factor in $H \cup F_1$. This contradicts the statement (*) and concludes our main theorem. See Fig. 10. ∎

By applying Lemma 3, the main theorem can be slightly improved.

COROLLARY 1. *Let G be a k-regular graph of order 2n and $n \leqslant k$. Then G contains at least $\lfloor n/2 \rfloor + (k-n)$ disjoint 1-factors.*

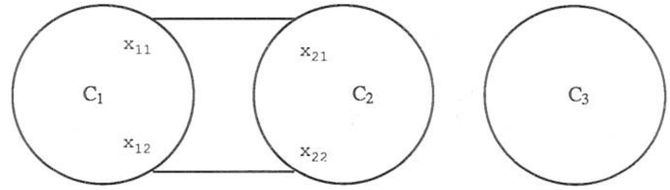

FIGURE 9

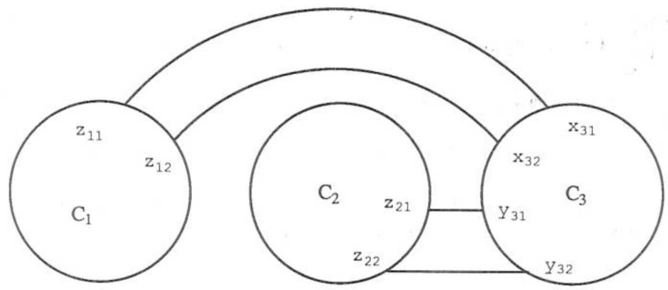

FIGURE 10

By applying Lemma 9, we have the following corollary:

COROLLARY 2. *Let G be a k-regular graph of order $2n$ and $n \leqslant k$. Let $p_1, ..., p_s$ be odd positive integers and $p_{s+1}, ..., p_r$ be even positive integers such that $p_1 + \cdots + p_r = k$. If*

$$s \leqslant \left\lfloor \frac{n}{2} \right\rfloor + (k-n),$$

then G is $(p_1, ..., p_r)$-factorizable

Note Added in Proof. Theorem B was recently improved by H. Li for large degree k (see [LH]).

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