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Small Circuit Double Covers of Cubic Multigraphs

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Let G be a two-connected graph. A family F of circuits of G is called a circuit double cover (CDC) if each edge of G is contained in exactly two circuits of F. In this paper, we show that if a simple cubic graph G ($G \neq K_4$) of order n has a CDC, then G has a CDC containing at most $n/2$ circuits. This result establishes the equivalence of the circuit double cover conjecture (due to Szekeres, Seymour) and the small circuit double cover conjecture (due to Bondy) for any cubic graph. Actually, a stronger result is obtained in this paper for all loopless cubic graphs. Another result in this paper establishes an upper bound on the size of any CDC of a cubic graph.

1. INTRODUCTION

We follow the terminology and notations of [BM]. Unless otherwise stated, the graphs considered in this paper are connected and loopless (parallel edges are allowed).

1.1. *Circuit Double Covers*

Let G be a connected cubic graph of order n. If G has a family F of circuits such that each edge of G is contained in exactly two circuits of F, then F is called a *circuit double cover* or, for short, a CDC, of G.

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The following conjectures are well known. The main result of this paper will establish their equivalence.

Conjecture A (Szekeres [SZ], Seymour [S], or see [J1, J2]). Every two-connected cubic graph has a circuit double cover.

Conjecture B (Bondy [B1]). Every two-connected simple cubic graph G of order n has a circuit double cover consisting of at most $n/2$ circuits if $G \neq K_4$.

In the following theorem, we establish an upper bound on the size of any CDC of a cubic graph.

THEOREM 1. *If F is a circuit double cover of a connected cubic graph G of order n, then $|F| \leqslant n/2 + 2$.*

1.2. *Small Circuit Double Covers*

A loopless cubic graph with two vertices and three parallel edges is denoted by $K_2^{(3)}$ and a complete graph with four vertices is denoted by K_4.

A connected graph with four vertices, two of which are of degree one and two of which are of degree three, is called a ϕ-graph (see Fig. 1). Let G be a loopless cubic graph. A *blistering* of G is constructed by recursively replacing edges by ϕ-graphs (see Fig. 2). For the sake of convenience, we say that a graph G is a blistering of itself (replacing edges by ϕ-graphs zero times). Figure 2 illustrates this concept with some examples: a blistered $K_2^{(3)}$ and a blistered K_4. (Note that this definition of a blistered graph is different from the definition originally given in [AGZ]).

A CDC F of a connected cubic graph G is called a *small circuit double cover* or, for short, an SCDC, of G, provided that

(i) $|F| \leqslant n/2 + 2$, if G is a blistered $K_2^{(3)}$;

FIGURE 1

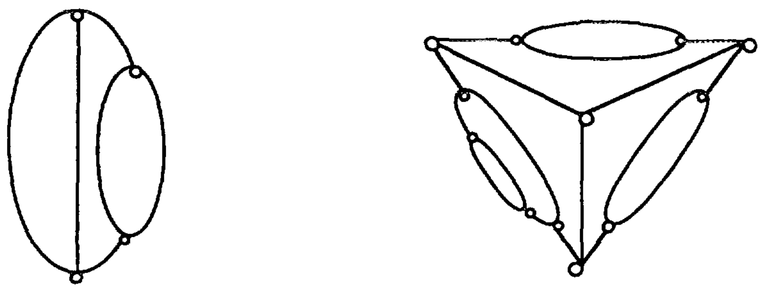


FIGURE 2

(ii) $|F| \leqslant n/2+1$, if G is a blistered K_4;

(iii) $|F| \leqslant n/2$, otherwise.

By the definition of blistered graphs, $G = K_2^{(3)}$ and $G = K_4$ are included in (i) and (ii), respectively. Note that the definition of a small circuit double cover is an extension of the original definition of SCDC introduced by Bondy [B1]. Let Γ_3 be the set of all two-connected cubic graphs, let Γ_{CDC} be the set of all connected cubic graphs admitting a CDC and Γ_{SCDC} be the set of all connected cubic graphs admitting an SCDC. Obviously,

$$\Gamma_{\mathrm{SCDC}} \subseteq \Gamma_{\mathrm{CDC}} \subseteq \Gamma_3.$$

The following problem is a refinement of Conjecture B.

Conjecture B′. Every two-connected cubic graph has a small circuit double cover (that is, $\Gamma_{\mathrm{SCDC}} = \Gamma_3$).

Previous Results [LYZ]. (i) If every two-connected cubic graph has a circuit double cover, then every two-connected cubic graph has a small circuit double cover (that is, if $\Gamma_{\mathrm{CDC}} = \Gamma_3$ then $\Gamma_{\mathrm{SCDC}} = \Gamma_3$).

(ii) Every two-connected cubic graph containing no subdivision of the Petersen graph has a small circuit double cover. (It was proved in [AZ] that every such graph has a circuit double cover.)

(iii) Every three-edge-colorable cubic graph has a small circuit double cover. (The case of hamiltonian cubic graphs was originally proved in [Y].)

Some related results about the small circuit double cover also can be found in [B1, B2, LH, SK], etc. The following problem was proposed in [LYZ] and is solved in this paper. One of the techniques that we use here is similar to one employed by Goddyn [G] in showing that the girth of a smallest counterexample to Conjecture A is at least seven.

THEOREM 2. *If a two-connected cubic graph G has a circuit double cover, then G has a small circuit double cover (that is, $\Gamma_{\mathrm{SCDC}} = \Gamma_{\mathrm{CDC}}$).*

1.3. *Strong Embedding of Cubic Graphs*

A graph is said to be *embedded* in a surface S (a closed two-manifold) if it can be drawn in S so that edges intersect only at their common vertices. If G is embedded in a surface S, then we regard G as a topological subspace of S and each component of $S \backslash G$ is called a face of the embedding. An embedding of G in S is a *strong-embedding* if every face is homeomorphic to the open disk and each face boundary is a circuit of G. (A strong embedding is also sometimes called a *circular embedding*, see [J1, J2].) As indicated by Jaeger [J1], when G is a cubic graph, every circuit double cover F is the system of face boundaries of a strong embedding in some surface S. The surface S is said to be induced by the CDC F. A recent result due to Richter, Seymour, and Širáň [RSS] asserts that every three-connected planar graph has a strong embedding in some non-spherical surface. For cubic graphs, the following corollary of Theorem 2 generalizes this result, assuming the truth of the CDC conjecture.

COROLLARY 3. *Every two-connected cubic simple graph G has a strong embedding in some non-spherical surface if and only if G has a* CDC.

Proof. Let F be an SCDC of G and let S be the surface induced by F. Denote the Euler characteristic of S by $k(S)$. Then by Euler's formula,

$$|V(G)| + |F| - |E(G)| = k(S).$$

Since G is cubic $|E(G)| = 3|V(G)|/2$ and by Theorem 2, $|F| \leqslant |V(G)|/2 + 1$, unless G is a blistered $K_2^{(3)}$. It follows that $k(S) \leqslant 1$ if G is simple. The surface S must thus be non-spherical. ∎

1.4. *Small circuit 2k-Covers of Cubic Graphs*

A two-edge-connected graph G is said to be *circuit 2k-coverable* if G has a family F of circuits such that each edge of G is contained in precisely $2k$ circuits of F. This family of circuits is called a *circuit 2k-cover* of G; when $k = 1$, we have a circuit double cover. Unlike the circuit double cover conjecture, which is still open, all other circuit $2k$-cover problems (for $k \geqslant 2$) have been solved. The circuit four-cover theorem is due to Bermond, Jackson, and Jaeger (see [BJJ]) and the circuit six-cover theorem is due to Fan (see [F]). As mentioned in [F], the existence of a circuit $2k$-cover (for $k \geqslant 2$) of any two-edge-connected graph is immediately implied by the above two results. The small circuit double cover conjecture for cubic graphs is verified in Theorem 2, assuming the existence of a circuit double cover. The result below generalizes Theorem 2 to $2k$-coverings. Because of the theorems of Bermond, Jackson, and Jaeger and Fan, the assumption of the existence of a $2k$-cover for a graph can be dropped. By imitating the proof of Theorem 2 and by dropping the assumption that G has a circuit double cover, we obtain the following theorem.

THEOREM 4. *Let G be a two-edge-connected cubic graph with n vertices, let $k \geq 2$ be an integer, and let $\mathrm{SC}_k(G)$ denote the number of circuits in a smallest circuit $2k$-cover of G. Then*

(i) $\mathrm{SC}_k(G) \leq k(n/2+2)$, *if G is a blistered $K_2^{(3)}$;*

(ii) $\mathrm{SC}_k(G) \leq k(n/2+1)$, *if G is a blistered K_4;*

(iii) $\mathrm{SC}_k(G) \leq k(n/2)$, *for all other graphs.*

2. Circuit Double Covers of Cubic Graphs

For any connected cubic graph G admitting a CDC, Theorem 2 establishes an upper bound on the size of a smallest CDC of G (a max–min problem), while the following theorem provides an upper bound for *all* CDCs of G.

THEOREM 1. *If F is a CDC of a connected cubic graph G of order n, then* $|F| \leq n/2+2$.

Proof. It is well known that the circuit space of a connected graph with n vertices and m edges has dimension $m-n+1$. The addition operation in this vector space is the symmetric difference (binary sum) of edge sets of the circuits. The CDC F is a subset of the circuit space of G. Hence the rank $r(F)$ of F (the maximum number of independent circuits in F) satisfies the inequality

$$r(F) \leq \frac{3n}{2} - n + 1 = \frac{n}{2} + 1.$$

Now we claim that $r(F) = |F| - 1$. For otherwise, there is a proper subset F' of F such that the binary sum $\sum_{C \in F'} E(C) = \emptyset$. The circuits of F' induce a proper subgraph H of G, and each edge of H is covered twice by F'. Let e be any edge of $G \backslash E(H)$ with at least one end in H. Since G is cubic, any circuit in G containing e must use at least one edge in H. This is a contradiction since e must be covered twice by the CDC F. Hence $|F| - 1 \leq n/2 + 1$, and so $|F| \leq n/2+2$. ∎

An Alternative Proof (L. Goddyn and B. Richter, personal communication). Each circuit of F can be considered as the boundary of a disk. The graph G is therefore embedded in a surface S established by joining all these disks at the edges of G. Since the Euler characteristic of S is not greater than two, by Euler's formula, we have that

$$|V(G)| + |F| - |E(G)| \leqslant 2.$$

Note that $|V(G)| = n$ and $|E(G)| = 3n/2$ since G is cubic. Therefore, no circuit double cover F of G contains more than $n/2 + 2$ circuits. ∎

Actually, the alternative proof gives a generalization of Theorem 1.

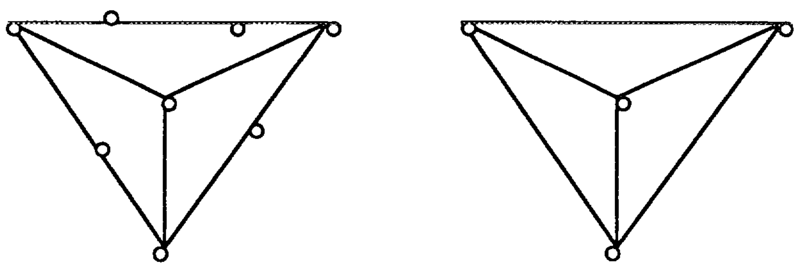
THEOREM 1′. *If F is a* CDC *of a connected cubic graph G of order n, then $|F| \leqslant n/2 + k(S)$, where S is the surface induced by F and $k(S)$ is the Euler characteristic of the surface S.*

3. Small Circuit Double Covers of Cubic Graphs

If G is a loopless graph in which the degree of each vertex is either two or three, then the cubic graph that is *homeomorphic* to G is called the *background graph* of G and is denoted by $B(G)$ (see Fig. 3). A *trivial cut* X of a graph G is an edge-cut of G such that one component of $G \setminus X$ is a single vertex.

THEOREM 2. *If a two-connected cubic graph G has a circuit double cover, then G has a small circuit double cover (that is, $\Gamma_{SCDC} = \Gamma_{CDC}$).*

Proof. Assume that $\Gamma_{SCDC} \neq \Gamma_{CDC}$. Let G be a smallest graph in $\Gamma_{CDC} \setminus \Gamma_{SCDC}$. Since $K_2^{(3)}$ and K_4 belong to Γ_{SCDC}, $G \neq K_2^{(3)}$, K_4. Let $|V(G)| = n$.



G B (G)

FIGURE 3

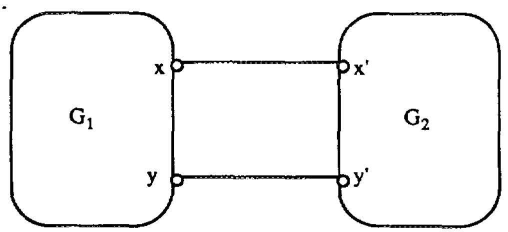


FIGURE I-1

I. *G has no two-cut.* Assume that G has a two-cut. Choose a two-cut $X=\{xx', yy'\}$, where G_1 and G_2 are two components of $G\backslash X$ and $x, y \in V(G_1)$, $x', y' \in V(G_2)$ such that G_1 is as small as possible. Note that $x \neq y$, $x' \neq y'$ since G is cubic and has no cut-edge (see Fig. I-1). Let $H_1 = G_1 \cup \{e\}$ and $H_2 = G_2 \cup \{e'\}$, where e and e' are new edges joining x and y, x' and y', respectively (see Fig. I-2). If G is not simple then, by the choice of X, $|V(G_1)| = 2$ and $H_1 = K_2^{(3)}$. Let F be a CDC of G. Let C_1 and C_2 be the two circuits of F containing xx' and yy'. Let C_i' be the segment of C_i of G_i between x and y, together with the edge e, $i = 1, 2$. Then

$$\{C \in F: C \neq C_1, C_2 \text{ and } E(C) \cap E(H_1) \neq \emptyset\} \cup \{C_1', C_2'\}$$

is a CDC of H_1. That is, $H_1 \in \Gamma_{\text{CDC}}$. By the inductive hypothesis, $H_1 \in \Gamma_{\text{SCDC}}$. Similarly, $H_2 \in \Gamma_{\text{SCDC}}$.

Let F_1^* and F_2^* be SCDCs of H_1 and H_2, respectively. Let D_1', D_1'' (respectively, D_2', D_2'') be the circuits of F_1^* (respectively, F_2^*) containing the new edge $e = xy$ (respectively, $e' = x'y'$).

Let $D' = D_1' \Delta D_2' \Delta C_4$ and $D'' = D_1'' \cup D_2'' \Delta C_4$, where C_4 is a circuit $xx'y'yx$ and Δ is the symmetric difference. Then

$$F^{**} = [F_1^* \cup F_2^* \cup \{D', D''\}] \backslash \{D_1', D_1'', D_2', D_2''\}$$

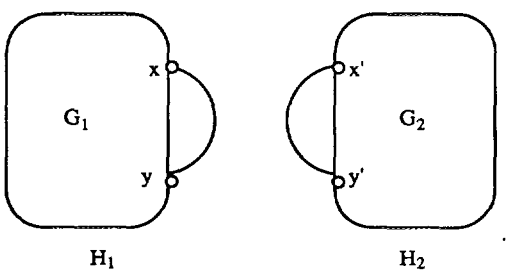


FIGURE I-2

is a CDC of G consisting of $|F_1^*|+|F_2^*|-2$ circuits. Note that $|V(G)|=|V(H_1)|+|V(H_2)|$, so F^{**} is an SCDC of G if $|F_1^*|+|F_2^*|\leqslant n/2+2$. Thus, by the assumption, we have that $|F_1^*|+|F_2^*|\geqslant n/2+3$. That is, one of H_1, H_2 must be a blistered $K_2^{(3)}$ and the other must be either a blistered $K_2^{(3)}$ or a blistered K_4. It is evident that any blistered $K_2^{(3)}$ has at least two distinct pairs of parallel edges. If H_i (for $i=1$ or 2) is a blistered $K_2^{(3)}$, then one pair of parallel edges of H_i must originally exist in G and therefore G is not simple. By the choice of the edge-cut X and the component G_1, we can see that $|V(G_1)|=2$ and $H_1=K_2^{(3)}$. Thus G is a blistered graph of H_2. Furthermore, G is a blistered $K_2^{(3)}$. (respectively, a blistered K_4) if H_2 is a blistered $K_2^{(3)}$ (respectively, a blistered K_4). Note that blistering one edge adds two vertices and requires exactly one more circuit to double cover the new edges. Therefore the CDC F^{**} constructed above is an SCDC of G. This is a contradiction. Thus G has no two-circuit and is *simple.*

II. *G has no non-trivial three-cut.* Suppose that G has a non-trivial three-cut $X=\{xx', yy', zz'\}$ with two non-trivial components G_1 and G_2 (see Fig. II-1). Since G is cubic and has no two-cut, X is a matching of G. Let H_1 (respectively, H_2) be the graph constructed from G by contracting all edges in G_2 (respectively, G_1), and denote the new vertex in H_1 (respectively, H_2) by w_2 (respectively, w_1) (see Fig. II-2). Let F be a CDC of G. Let C_{xy} be the circuit of F containing the edges xx' and yy'; the circuits C_{xz} and C_{yz} of F are defined similarly. Let C'_{xy} (respectively, C'_{xz} and C'_{yz}) be the circuit constructed from C_{xy} (respectively, C_{xz} and C_{yz}) by contracting all edges in G_2. Then

$$\{C\colon C\in F \text{ and } E(C)\subseteq E(G_1)\}\cup\{C'_{xy}, C'_{xz}, C'_{yz}\}$$

is a CDC of H_1. By the inductive hypothesis, $H_1\in\Gamma_{\text{SCDC}}$. Similarly, $H_2\in\Gamma_{\text{SCDC}}$. Let F_1^* and F_2^* be SCDCs of H_1 and H_2, respectively. Let D'_{xy} be the circuit of F_1^* containing the edges xw_2 and yw_2; define D'_{xz}, D'_{yz} similarly. Let D''_{xy} be the circuit of F_2^* containing the edges $x'w_1$ and $y'w_1$; define D''_{xz}, D''_{yz} similarly. Let

$$D_{xy}=[D'_{xy}\cup D''_{xy}\cup\{xx', yy'\}]\setminus\{xw_2, yw_2, x'w_1, y'w_1\};$$

define D_{xz}, D_{yz} similarly. Then

$$F^{**}=[F_1^*\cup F_2^*\cup\{D_{xy}, D_{xz}, D_{yz}\}]\setminus\{D'_{xy}, D'_{xz}, D'_{yz}, D''_{xy}, D''_{xz}, D''_{yz}\}$$

is a CDC of G. Note that G is simple and X is a three-matching. Thus both H_1 and H_2 are simple and neither H_1 nor H_2 is a blistered $K_2^{(3)}$. Therefore, by the inductive hypothesis,

$$|F_1^*|\leqslant\frac{|V(H_1)|}{2}+1, \qquad |F_2^*|\leqslant\frac{|V(H_2)|}{2}+1.$$

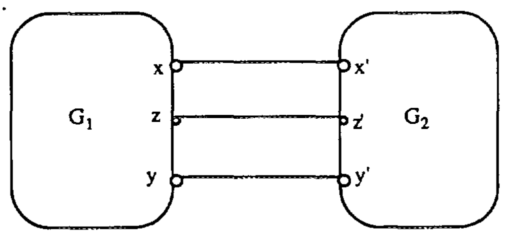


FIGURE II-1

Since $|F^{**}| = |F_1^*| + |F_2^*| - 3$ and $|V(G)| = |V(H_1)| + |V(H_2)| - 2$, F^{**} is an SCDC of G, a contradiction.

Hence we can see that G is *triangle-free*.

III. *No CDC of G contains a four-circuit.* Let F be a CDC of G. Assume that F has some circuit of length four, say $C = uvwx$. Let $u'u$, $v'v$, $w'w$, and $x'x$ be four edges of $E(G)\backslash E(C)$. Note that since G has no three-circuit, either $\{u'u, v'v, w'w, x'x\}$ is a four-matching or (without loss of generality) $u' = w'$. If $u' = w'$, then G has a three-cut consisting of vv', xx' and the edge incident with $u' = w'$ other than uu' and ww'. Since G has no non-trivial three-cut, we have that $v' = x'$ and therefore $G = K_{3,3}$ for which the theorem holds. So we assume that $\{u'u, v'v, w'w, x'x\}$ is a four-matching of G. Let C_1 be the circuit of F containing $u'uvv'$, C_2 be the circuit of F containing $v'vww'$, C_3 be the circuit of F containing $w'wxx'$, and C_4 be the circuit of F containing $x'xuu'$ (see Fig. III-1).

Case 1. $C_2 \neq C_4$ (or, symmetrically, $C_1 \neq C_3$). Let $D = C_4 \Delta C$. Then $[F\backslash\{C_4, C\}] \cup \{D\}$ is a CDC of $H = G\backslash\{ux\}$ (see Fig. III-2). Since the background graph $B(H) \in \Gamma_{CDC}$, by the inductive hypothesis, $B(H) \in \Gamma_{SCDC}$. Let F^* be an SCDC of $B(H)$. Since G is triangle-free, $B(H)$ is simple and not a blistered graph. Furthermore, $B(H)$ is neither $K_2^{(3)}$ nor K_4 because $B(H)$ contains at least six vertices $\{u', v, v', w, w', x'\}$. Thus F^* consists of at most $|V(B(H))|/2 = (n-2)/2$ circuits.

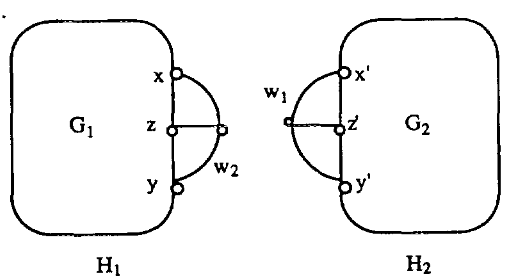


FIGURE II-2

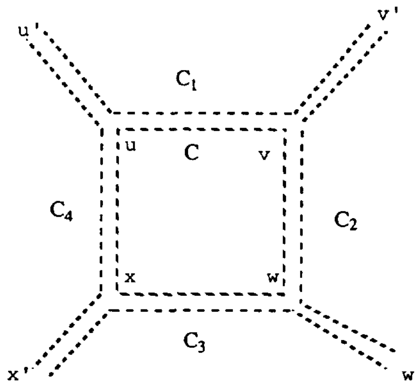


FIGURE III-1

Subcase 1. F^* has a circuit D_1 containing the path $u'uvwxx'$ (see Fig. III-3). Let $D_2 = D_1 \Delta C$. Then $[F^* \setminus \{D_1\}] \cup \{D_2, C\}$ is an SCDC of G, a contradiction.

Subcase 2. The path $u'uvwxx'$ does not belong to any circuit of F^*. Then the circuits of F^* containing v or w must be of the following four types E_1, E_2, E_3, E_4 (see Fig. III-4): E_1 contains $u'uvww'$, E_2 contains $u'uvv'$, E_3 contains $x'xwvv'$, and E_4 contains $x'xww'$.

(i) If $E_2 \neq E_4$, let $E'_2 = E_2 \Delta C$, $E'_3 = E_3 \Delta C$ (note that $E_2 \neq E_3$ because $E_2 \cup E_3$ has a vertex of degree three). Then (see Fig. III-5) $[F^* \setminus \{E_2, E_3\}] \cup \{E'_2, E'_3\}$ is an SCDC of G.

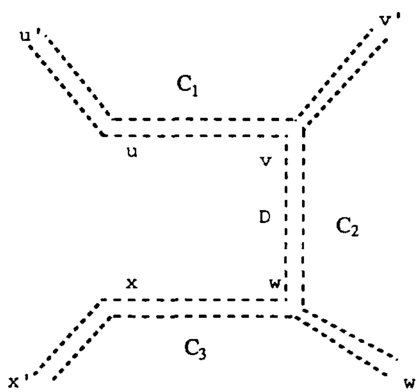


FIGURE III-2

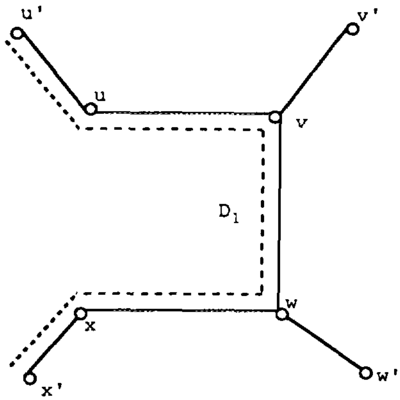


FIGURE III-3

(ii) If $E_2 = E_4$, then the union of the circuit E_2 and its chord ux can be covered by two circuits E_5 and E_6 such that $E_5 \Delta E_6 = E_2$ and $E_5 \cap E_6 = \{ux\}$. Thus $[F^* \setminus \{E_2\}] \cup \{E_5, E_6\}$ is an SCDC of G, a contradiction.

Case 2. $C_2 = C_4$ and $C_1 = C_3$ (refer to Fig. III-1). We claim that either $u'v', w'x' \notin E(G)$ or $v'w', x'u' \notin E(G)$. Without loss of generality, assume to the contrary that $v'w', w'x' \in E(G)$. Then the edges $\{uu', v'v'', x'x''\}$, where $v'' \in N(v') \setminus \{v, w'\}$, $x'' \in N(x') \setminus \{x, w'\}$, form a three-edge-cut of G (see Fig. III-6). Thus G is a three-cube since G has no non-trivial three-cut. It is very easy to check that the theorem holds for the three-cube.

Suppose that $u'v', w'x' \notin E(G)$. Then the background graph of $H' = G \setminus \{ux, vw\}$ is simple. Let $D = C_4 \Delta C$, a union of circuits. Then

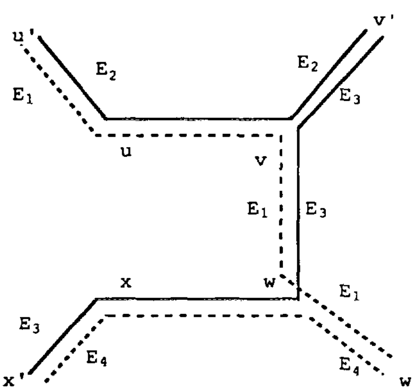


FIGURE III-4

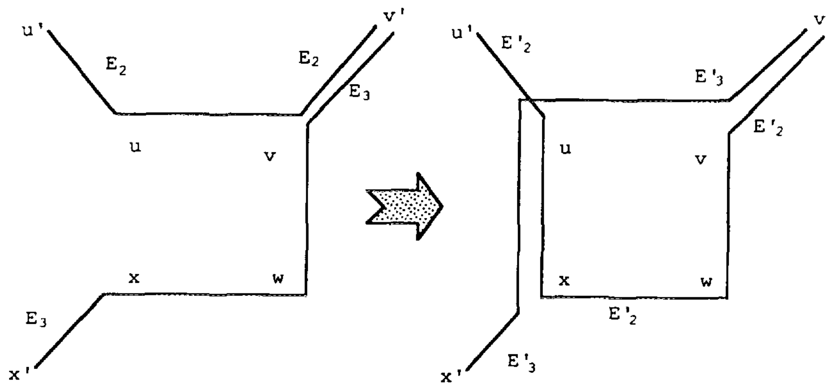


FIGURE III-5

$[F \backslash \{C_4, C\}] \cup \{D\}$ is a CDC of $H' = G \backslash \{ux, vw\}$. By the inductive hypothesis, the background graph $B(H') \in \Gamma_{\text{SCDC}}$. Let F^{**} be an SCDC of $B(H')$. Note that $B(H')$ contains at least four vertices $\{u', v', w', x'\}$. If $B(H') = K_4$, then the graph G is illustrated in Fig. III-7; an SCDC can easily be found in this graph. Since $B(H)$ is simple, we may thus assume that $B(H')$ is neither a blistered $K_2^{(3)}$ nor a blistered K_4. Hence $|F^{**}| \leqslant |V(B(H'))|/2 = n/2 - 2$. Let D_1, D_2 be the circuits of F^{**} containing the path $u'uvv'$ *and* D_3, D_4 be circuits of F^{**} containing the path $w'wxx'$ (see Fig. III-8).

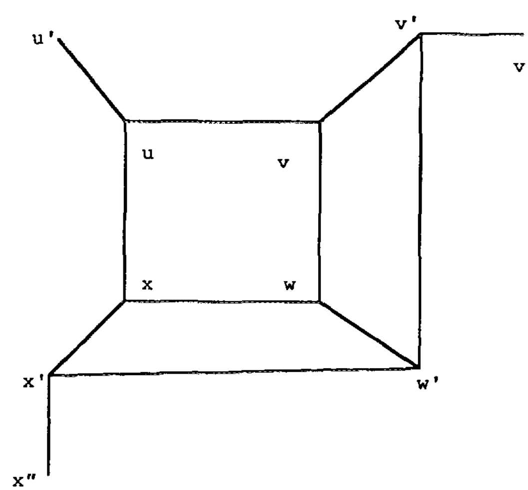


FIGURE III-6

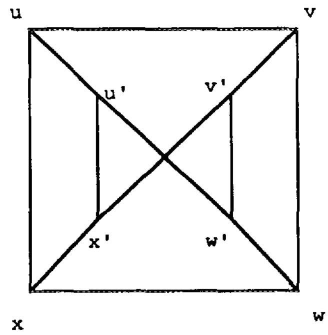


FIGURE III-7

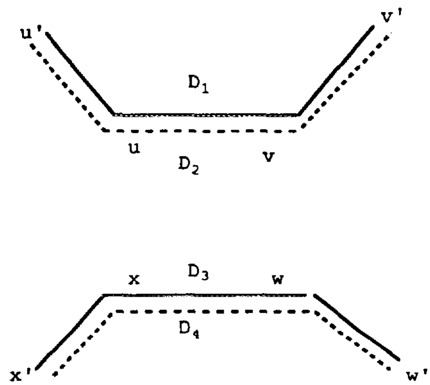


FIGURE III-8

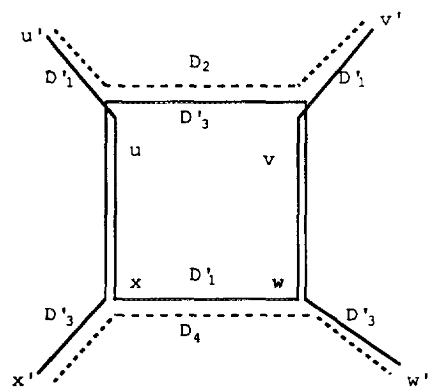


FIGURE III-9

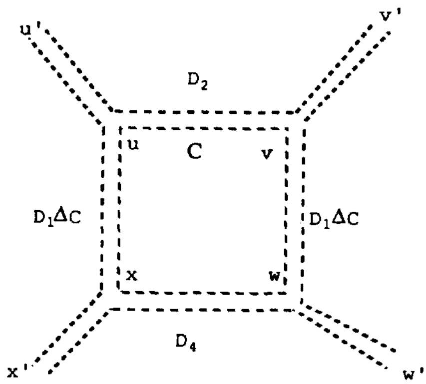


FIGURE III-10

Subcase 1. $\{D_1, D_2\} \cap \{D_3, D_4\} = \emptyset$. Let $D_1' = D_1 \Delta C$ and $D_3' = D_3 \Delta C$ (see Fig. III-9). Then $[F^{**} \setminus \{D_1, D_3\}] \cup \{D_1', D_3'\}$ is an SCDC of G, a contradiction.

Subcase 2. $\{D_1, D_2\} \cap \{D_3, D_4\} \neq \emptyset$. Without loss of generality, suppose that $D_1 = D_3$. The symmetric difference of D_1 and C is the union of at most two circuits since $D_1 \setminus C$ has only two segments. Thus $[F^{**} \setminus \{D_1\}] \cup \{D_1 \Delta C, C\}$ (see Fig. III-10) is an SCDC of G consisting of at most $n/2$ circuits.

IV. *No CDC of G contains a five-circuit.* Assume that the CDC F of G contains a circuit C of length five. Let $C = x_1 x_2 \cdots x_5 x_1$ and y_i be the neighbor of x_i other than x_{i-1} and x_{i+1} (mod 5). Since G is triangle-free,

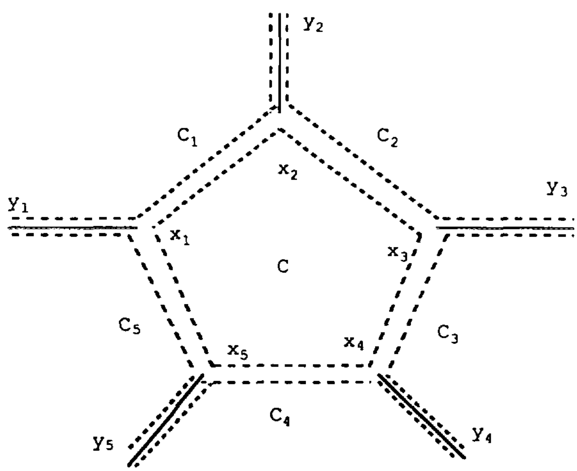


FIGURE IV-1

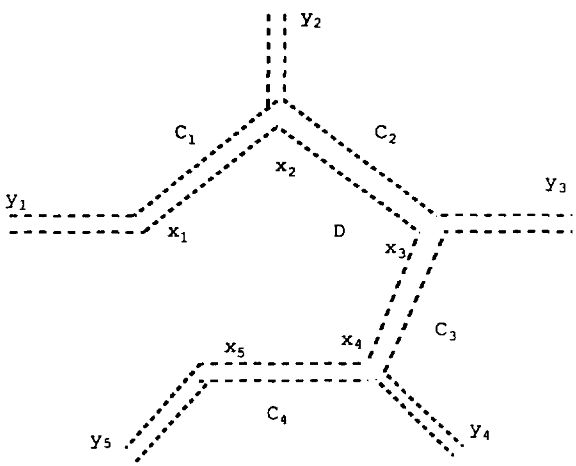


FIGURE IV-2

$\{x_1, ..., x_5\}$ and $\{y_1, ..., y_5\}$ are disjoint. Denote the circuit of F containing the path $y_i x_i x_{i+1} y_{i+1}$ (mod 5) by C_i (see Fig. IV-1).

(i) Since F is a CDC of G, $C_i \neq C_{i \pm 1}$ for $i = 1, ..., 5$ (mod 5). Hence $\{C_1, ..., C_5\}$ is a set of at least three distinct circuits, and one element of it must be distinct from all others.

(ii) By (i), we assume, without loss of generality, that $C_5 \neq C_1$, C_2, C_3, and C_4. Let $D = C_5 \Delta C$ (see Fig. IV-2). Then $[F \backslash \{C_5, C\}] \cup \{D\}$ is a CDC of $H = G \backslash \{x_1 x_5\}$. By the inductive hypothesis, $B(H) \in \Gamma_{SCDC}$. Let F^* be an SCDC of $B(H)$.

(iii) We claim that $|F^*| \leqslant |V(B(H))|/2$. By the inductive hypothesis, we only need to show that $B(H)$ is a simple graph other than $K_2^{(3)}$ and K_4. Since G is triangle-free, $B(H)$ must be simple and $|\{y_1, ..., y_5\}| \geqslant 3$. Thus $B(H)$ has at least six distinct vertices $(x_2, x_3, x_4, y_1, ..., y_5\}$. This excludes the possibility that $B(H) = K_4$, so our claim holds.

(iv) Since $\{x_i y_i\colon 1 \leqslant i \leqslant 5\}$ is an edge-cut, every circuit in F^* contains an even number of edges in $\{x_i y_i\colon 1 \leqslant i \leqslant 5\}$ and so the edge set $\{x_i y_i\colon 1 \leqslant i \leqslant 5\}$ of H is covered by at most five distinct circuits of F^*. Let D_1 and D_2 be the circuits of F^* containing the path $y_5 x_5 x_4$ and let E_1 and E_2 be the circuits of F^* containing $y_1 x_1 x_2$.

We claim that D_1, D_2, E_1, E_2 are distinct. It is trivial that $D_1 \neq D_2$ and $E_1 \neq E_2$. Assume that $D_1 = E_1$. Then the union of the circuit D_1 and its chord $x_5 x_1$ can be covered by two circuits D' and D'' such that $D' \cap D'' = \{x_5 x_1\}$ and $D' \Delta D'' = D_1$. Thus $[F^* \backslash \{D_1\}] \cup \{D', D''\}$ is an SCDC of G. This contradicts the assumption that G is a counterexample to the theorem.

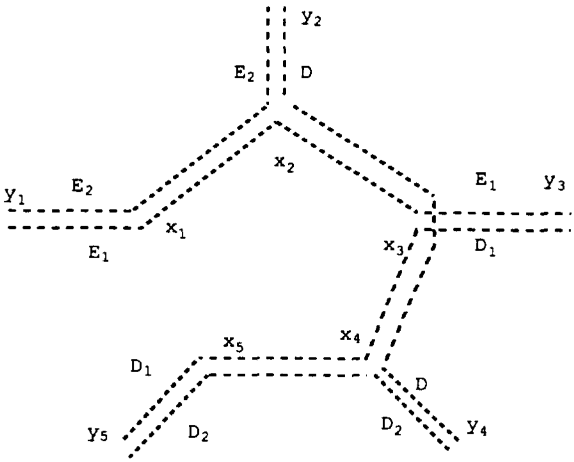


FIGURE IV-3

(v) For $i = 1, 2$, let D_i contain the path $y_5 x_5 x_4 \cdots x_{d_i} y_{d_i}$ for some $d_i \in \{2, 3, 4\}$ (note $d_i \neq 1$ by (iv)) and let E_i contain the path $y_1 x_1 x_2 \cdots x_{e_i} y_{e_i}$ for some $e_i \in \{2, 3, 4\}$. It can be seen that $d_1 \neq d_2$, for otherwise the edges $x_{d_1} y_{d_1}$, $x_{d_1+1} x_{d_1}$ are covered twice by the circuits D_1, D_2 and the edge $x_{d_1} x_{d_1-1}$ cannot be covered by any circuit of F^*. Similarly, $e_1 \neq e_2$. Since d_1, d_2, e_1, $e_2 \in \{2, 3, 4\}$, we assume, without loss of generality, that $d_1 = e_1$.

(vi) *Case* 1. $d_1 = e_1 = 3$. The coverage of all edges incident with $x_1, ..., x_5$ by F^* in H can be easily determined and is illustrated in Fig. IV-3. The circuit D in Fig. IV-3 contains the path $y_2 x_2 x_3 x_4 y_4$. Obviously D is distinct from each of D_1, D_2, E_1, and E_2 since it intersects all of them. Let $D_1' = D_1 \Delta C$ and $E_1' = E_1 \Delta C$. Then $[F^* \setminus \{D_1, E_1\}] \cup \{D_1', E_1'\}$ is an SCDC of G, a contradiction.

(vii) *Case* 2. $d_1 = e_1 = 2$ (or, symmetrically, $d_1 = e_1 = 4$). The coverage of all edges incident with $x_1, ..., x_5$ by F^* in H is illustrated in Fig. IV-4. The circuit D in Fig. IV-4 contains the path $y_3 x_3 x_4 y_4$. Obviously the circuit D is distinct from each of D_1, D_2, and E_2, while it is possible that $D = E_1$. As in Case 1, let $D_1' = D_1 \Delta C$ and $E_1' = E_1 \Delta C$. If $D \neq E_1$, then $[F^* \setminus \{D_1, E_1\}] \cup \{D_1', E_1'\}$ is an SCDC of G, a contradiction. Assume that $D = E_1$. Then $E_1' = E_1 \Delta C = [E_1 \setminus \{x_1 x_2, x_3 x_4\}] \cup \{x_2 x_3, x_4 x_5, x_5 x_1\}$. Thus $F^{**} = [F^* \setminus \{D_1, E_1\}] \cup \{D_1', E_1'\}$ is a CDC of $H' = G \setminus \{x_3 x_4\}$ (see Fig. IV-5). Here $E_1' = E_1 \Delta C$ is the union of at most two circuits. And $|F^{**}| = |F^*| \leqslant (n-2)/2$ if E_1' is a single circuit, or $|F^{**}| = |F^*| + 1 \leqslant n/2$ if E_1' is the union of two disjoint circuits.

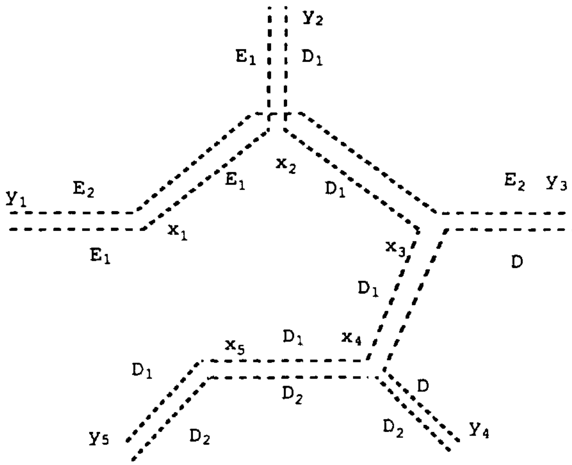


FIGURE IV-4

(a) If E_1' is the union of two disjoint circuits E^* and E^{**}, where E^* contains the path $y_1x_1x_5x_4y_4$ and E^{**} contains the path $y_2x_2x_3y_3$, let $D^* = E^* \Delta C$ and $D^{**} = E_2 \Delta C$. Then $[F^{**} \setminus \{E^*, E_2\}] \cup \{D^*, D^{**}\}$ is an SCDC of G.

(b) If E_1' is a single circuit, then the union of the circuit E_1' and its chord x_3x_4 can be covered by two circuits D° and $D^{\circ\circ}$ such that $D^\circ \cap D^{\circ\circ} = \{x_3x_4\}$ and $D^\circ \Delta D^{\circ\circ} = E_1'$, and $[F^{**} \setminus \{E_1'\}] \cup \{D^\circ, D^{\circ\circ}\}$ is an SCDC of G. Both contradict the assumption that G has no SCDC.

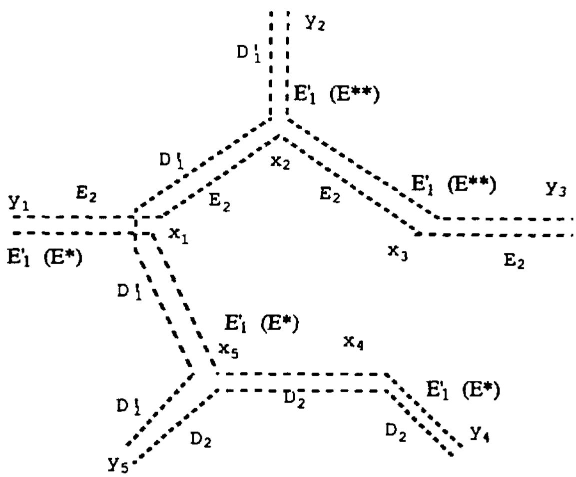


FIGURE IV-5

V. Note that the number of edges of the cubic graph G is $3n/2$, so the total length of all circuits of F is $3n$. That the length of each circuit of F is at least six implies that $|F| \leqslant n/2$. This is a contradiction and completes the proof of this theorem. ▮

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