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Cycle covers (III) – Compatible circuit decomposition and K_5-transition minor ☆

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A B S T R A C T

Let G be a 2-connected eulerian graph. For each vertex $v \in V(G)$, let $\mathcal{T}(v)$ be the set of edge-disjoint edge-pairs of $E(v)$, and, $\mathcal{T} = \bigcup_{v \in V(G)} \mathcal{T}(v)$. A circuit decomposition $\mathcal{C}$ of G is compatible with $\mathcal{T}$ if $|E(C) \cap P| \leq 1$ for every member $C \in \mathcal{C}$ and every $P \in \mathcal{T}$. Fleischner (1990's) wondered implicitly whether *if* $(G, \mathcal{T})$ *does not have a compatible circuit decomposition then* $(G, \mathcal{T})$ *must have an undecomposable* K_5*-transition-minor or its generalized transition-minor.* This long-standing open problem was partially verified for various *graph-minor-free* families of graphs, for example, it was solved by Fleischner for planar graphs (Fleischner (1980) [7]) and solved by Fan and Zhang for K_5-minor-free graphs (Fan and Zhang (2000) [6]). This *transition*-minor-free conjecture is now completely solved in this paper. And, as a by-product and a necessary stepping-stone, we characterize the structure of sup-undecomposable K_5-minor-free graphs $(G, \mathcal{T})$ in which every compatible circuit decomposition consists of a pair of Hamiltonian circuits. This result plays an important role in

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the proof of the main theorem and also generalizes an earlier result by Lai and Zhang (Lai and Zhang (2001) [13]).

1. Introduction

Compatible Circuit Decomposition (CCD) Problem. *Let G be a 2-connected eulerian graph with $\delta(G) \geq 4$, and for each $v \in V(G)$ let $\mathcal{T}(v)$ be a set of edge-disjoint edge-pairs (called transitions) of $E(v)$ (in the case of a loop l we allow $\{l, l\}$ to be a transition). Can we find a circuit decomposition $\mathcal{C}$ of G such that, for every $C \in \mathcal{C}$ and every $v \in V(G)$ and every $P \in \mathcal{T}(v)$, $|E(C) \cap P| \leq 1$ (unless C is a loop and $P = \{l, l\}$, in which case there is no* CCD*)?*

Such $\mathcal{C}$ is called compatible with the transition system $\mathcal{T} = \bigcup_{v \in V(G)} \mathcal{T}(v)$ (see also Definition 2.2).

The compatible circuit decomposition (CCD) problem is closely related to the famous circuit double cover conjecture, [12,14,16,17], and to the Sabidussi conjecture [7,8,9].

It is well known that not every eulerian graph associated with a transition system has a compatible circuit decomposition. For example, an undecomposable K_5 (or, *a bad* K_5 to use a more colloquial expression) is the complete graph K_5 associated with the transition system

$$\mathcal{T}_5 = \{\{v_i v_{i+\mu}, v_i v_{i-\mu}\} : \ i \in \mathbb{Z}_5, \ \mu \in \{1, 2\}\}$$

where $V(K_5) = \{v_0, v_1, \ldots, v_4\}$ (see Fig. 1).

The compatible circuit decomposition problem has been verified for planar graphs by Fleischner [7], and for K_5-minor-free graphs by Fan and Zhang [6]. Fleischner further asked implicitly the following question [10] which is beyond a graph-minor problem. In what follows we restrict ourselves to 2-connected graphs.

Problem 1 *(Fleischner [10]).* If $(G, \mathcal{T})$ does not have a compatible circuit decomposition, does $(G, \mathcal{T})$ contain either an undecomposable K_5-transition-minor or one of its generalized transition-minors?

A transition-minor is not only a graph-minor that preserves some topological structure of G but also inherits the original transition system $\mathcal{T}$ (see Definitions 2.8 and 2.10 for definitions of transition-minor and SUD-K_5). Problem 1 is completely solved in this paper.

Theorem 1. *Let $(G, \mathcal{T})$ be a 2-connected eulerian graph with the minimum degree $\delta \geq 4$ associated with a transition system. If $(G, \mathcal{T})$ is* SUD-K_5*-minor-free, then it has a compatible circuit decomposition.*

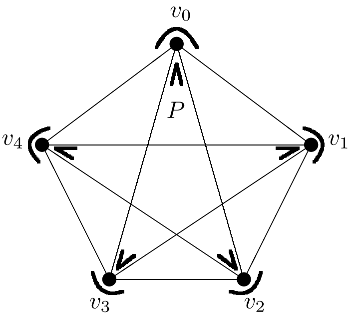

Fig. 1. K_5 with $\mathcal{T}_5 = \{\{v_{i-1}v_i, v_iv_{i+1}\}, \{v_{i-2}v_i, v_iv_{i+2}\} : i \in \mathbb{Z}_5\}$.

We observe that if $\mathcal{T} = \emptyset$, then any circuit decomposition of $(G, \mathcal{T})$ is in accordance with Theorem 1. Thus, we assume that our point of departure is a $(G, \mathcal{T})$ with $\mathcal{T} \neq \emptyset$.

In the study of circuit cover and circuit decomposition problems, one of the fundamental steps is to determine the structure of two adjacent circuits (i.e., two circuits having at least one vertex in common). The Hamilton weight problem ([13,19]) is one of such approaches for faithful cover problem. Its corresponding version for circuit decomposition is the Hamilton transition problem. That is, if $(G, \mathcal{T})$ has some compatible circuit decomposition and every such decomposition consists of a pair of hamiltonian circuits, then $(G, \mathcal{T})$ must be constructed recursively from two loops $(2L)$ via a series of $(X \leftrightarrow O)$-operations (the operation extending a vertex to a digon); see Definition 2.15 and Conjecture A. The family of transitioned graphs constructed in such a way is denoted by $\langle 2L \rangle$. This problem is solved in this paper for SUD-K_5-minor-free graphs, as stated in Theorem 2 below.

Theorem 2. *Let $(G, \mathcal{T})$ be a 4-regular fully transitioned graph such that it has some compatible circuit decomposition and every such decomposition consists of a pair of hamiltonian circuits. If $(G, \mathcal{T})$ is* SUD-K_5*-minor-free, then $(G, \mathcal{T}) \in \langle 2L \rangle$.*

This result plays a key role in the determination of a UD-K_5-transition-minor in Theorem 1. It is important to point out that both Theorems 1 and 2 are proved simultaneously because one provides the structures of extreme cases, while the other assures the existence of a compatible circuit decomposition for any proper minor of a smallest counterexample.

The rest of the paper is organized as follows. Some notation and terminology are recalled and introduced in Section 2. Main results, Theorems 1 and 2 are further summarized in Section 3. In Section 4, some preliminary lemmas for Theorem 1 are proved in Subsection 4.1 before its simultaneous proof with Theorem 2 (in Section 5). There are other important results (Lemmas 4.15 and 4.16) in Subsection 4.2 that determine the specific structure of UD-K_5 and is used in the simultaneous proof of Theorems 1 and 2.

2. Preliminary discussions

2.1. Basic definitions

For terminology and notation not defined here we follow [3,4,18], and the papers listed in the References.

A circuit is a 2-regular connected subgraph of a given graph G. A subgraph H of G is even if $\deg_H(v)$ is even for every vertex $v \in V(H)$.

Let v be a degree two vertex of a given graph G. Suppressing v is the operation of removing v and adding an edge between the two neighbours of v in G.

Definition 2.1. A vertex subset U is a separator of G separating G to G_1, G_2 if $E(G) = E(G_1) \cup E(G_2)$ and $V(G_1) \cap V(G_2) = U$ and $E(G_1) \cap E(G_2) = \emptyset$. U is a t-separator if $|U| = t$. We say a separator U separating subgraphs X_1, X_2 of G if U is a separator of G separating G to G_1, G_2 with $X_i \subseteq G_i$, $i = 1, 2$.

2.2. Transition system and CCD

Definition 2.2. Let G be an eulerian graph, and, for each $v \in V(G)$ with $\deg(v) \geq 4$, let $\mathcal{T}(v)$ be a set of edge-disjoint edge-pairs of $E(v)$. The set $\mathcal{T} = \bigcup_{v \in V(G)} \mathcal{T}(v)$ is called a transition system of G and each member of $\mathcal{T}$ is called a transition. A non-trivial vertex is a vertex with some transition (that is, $\mathcal{T}(v) \neq \emptyset$); otherwise, we called v a trivial vertex. The graph G with a transition system $\mathcal{T}$ is called a transitioned graph and denoted by $(G, \mathcal{T})$; (possibly $\mathcal{T} = \emptyset$). A fully transitioned graph is a transitioned graph without trivial vertex. For every subgraph H of G, $\mathcal{T}|_H = \{P \in \mathcal{T} \,|P \subset E(H)\}$. In the case of multiple edges e, f at $u, v \in V(G)$, we distinguish between the transition $\{e, f\}$ at u and the transition $\{e, f\}$ at v.

Definition 2.3. Let $(G, \mathcal{T})$ be a transitioned graph.

(1) A 1-separator $\{v\}$ separating G to G_1, G_2 is a bad cut-vertex if $E(v) \cap E(G_i) \in \mathcal{T}$ for at least one $i \in \{1, 2\}$.
(2) $(G, \mathcal{T})$ is admissible if it does not have a bad cut-vertex.

Definition 2.4. Let $(G, \mathcal{T})$ be a transitioned graph. Let $C = v_0 v_1 \ldots v_{r-1} v_0$ be a circuit. Let e_i be the edge of C joining v_i and v_{i+1} for every $i \in \mathbb{Z}_r$.

(1) v_i is an inner vertex of C if $\{e_{i-1}, e_i\} \in \mathcal{T}(v_i)$ or $E(v_i) \backslash \{e_{i-1}, e_i\} \in \mathcal{T}(v_i)$, and we call $\{e_{i-1}, e_i\}$ an inner transition of C at v_i. C is compatible at v_i if it is not an inner vertex of C.
(2) C is a compatible circuit of $(G, \mathcal{T})$ if C is compatible at every vertex of C.

Definition 2.5. A family $\mathcal{F}$ of circuits of G is a compatible circuit decomposition (abbreviated CCD) of $(G, \mathcal{T})$ if $\mathcal{F}$ is a circuit decomposition of G and every member of $\mathcal{F}$ is a compatible circuit.

It is obvious that the absence of bad cut-vertices (see Definition 2.3) is a necessary condition for a transitioned graph admitting a CCD.

Observation 2.6. *Consider a non-trivial vertex v of degree 4 in $(G, \mathcal{T})$. Let $E(v) = \{e_1, \ldots, e_4\}$ and $P = \{e_1, e_2\} \in \mathcal{T}(v)$. Then every circuit of a* CCD *of $(G, \mathcal{T})$ covers at most one edge of $\{e_3, e_4\}$. This means in a natural way and without loss of generality, we can assume that if $P \in \mathcal{T}(v)$, then $E(v) \setminus P \in \mathcal{T}(v)$, for every vertex v of degree 4. Thus every vertex v of degree 4 is either a trivial vertex, or $|\mathcal{T}(v)| = 2$.*

Definition 2.7. A circuit C is a removable circuit of $(G, \mathcal{T})$ if it is compatible and $(G \setminus E(C), \mathcal{T}|_{G \setminus E(C)})$ remains admissible (that is, $(G \setminus E(C), \mathcal{T}|_{G \setminus E(C)})$ has no bad cut-vertex).

Definition 2.8. Let $(G, \mathcal{T})$ be a transitioned eulerian graph, and, $G' = (G \setminus F_d)/F_c$ be an eulerian minor of G obtained by deleting F_d and contracting F_c where $F_d, F_c \subseteq E(G)$. The resulting transition system $\mathcal{T}' = \mathcal{T}|_{G'}$ on G' is defined as follows.

(1) Delete the edges of $(F_d \cup F_c)$. The resulting transition system $\mathcal{T}'$ contains all transitions $P \in \mathcal{T}$ for which $P \subseteq E(G \setminus (F_d \cup F_c))$.
(2) For each edge $e = v'_e v''_e \in F_c$, identify the end-vertices v'_e and v''_e as a new vertex v_e.
(3) Since we do not define a transition at any vertex v of degree 2, $\mathcal{T}'(v) = \emptyset$ if $\deg_{G'}(v) = 2$. And we apply Observation 2.6 to extend $\mathcal{T}'(z)$ if $\deg_{G'}(z) = 4$.

The resulting transitioned graph $(G', \mathcal{T}')$ is called a transition-minor of $(G, \mathcal{T})$.

Definition 2.9. $(G, \mathcal{T})$ is called the undecomposable K_5 (UD-K_5 for short) if $G = K_5$, and the transition system $\mathcal{T}$ is defined as follows.

$$\mathcal{T}(v_i) = \{\{v_i v_{i+\mu}, v_i v_{i-\mu}\} : \ \mu \in \{1, 2\} \pmod 5\}$$

for every $v_i \in V(K_5) = \{v_0, v_1, \ldots, v_4\}$; see Fig. 1.

Definition 2.10. The transitioned graph $(G, \mathcal{T})$ is a sup-undecomposable K_5 (SUD-K_5 for short) if the graph G can be decomposed into 15 connected edge-disjoint subgraphs

$$\{P_{i,j} : \ \{i, j\} \subset \mathbb{Z}_5, \ i < j\} \ \cup \ \{Q_i : \ i \in \mathbb{Z}_5\}$$

as follows (see Fig. 2).

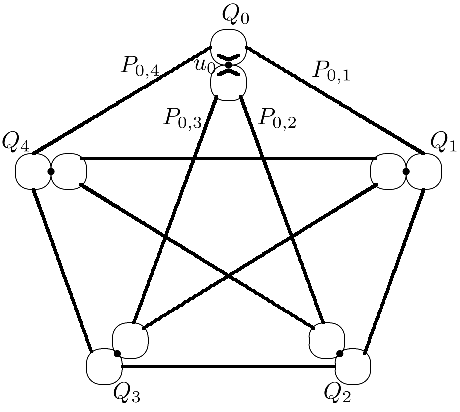

Fig. 2. A sup-undecomposable K_5.

(1) Each $P_{i,j}$ is a path joining $V(Q_i)$ and $V(Q_j)$ $(i < j)$, and the different $P_{i,j}$'s are internally disjoint;
(2) $\{Q_i : \ i \in \mathbb{Z}_5\}$ are disjoint connected subgraphs;
(3) Let Q_i^+ be the subgraph of H induced by $E(Q_i)$ and the four adjacent paths $P_{i,j}$ (for every pair $j \neq i$). Then each subgraph Q_i^+ has a bad cut-vertex u_i separating $P_{i,(i+1)} \cup P_{i,(i-1)}$ and $P_{i,(i+2)} \cup P_{i,(i-2)}$, where $u_i \in V(Q_i)$.

Note that a UD-K_5 is a special case of a SUD-K_5 where $|Q_i| = 1$ for every $i \in \mathbb{Z}_5$.

Definition 2.11. $(G, \mathcal{T})$ is sup-undecomposable K_5-transition-minor free (or, SUD-K_5-minor-free for short) if it does not have any eulerian minor H such that $(H, \mathcal{T}|_H)$ is a SUD-K_5.

The following is a straightforward observation.

Observation 2.12. *Let G' be an eulerian minor of G. If $(G, \mathcal{T})$ is* SUD-K_5*-minor-free, then $(G', \mathcal{T}')$ remains* SUD-K_5*-minor-free (where $\mathcal{T}'$ is described in Definition 2.8).*

Example 2.1. In [11], an infinite family of snarks $\{H_n\}$ has been constructed, which has a 2-factor F_n such that F_n is not contained in any circuit double cover of H_n. Let $\overline{H_n}$ be the 4-regular graph obtained from H_n by contracting the 1-factor $H_n \setminus F_n$ and $\mathcal{T}_n$ be the transition system of $\overline{H_n}$ such that each circuit of F_n has all its vertices as inner vertices (see Definition 2.4-(1)). Clearly, $(\overline{H_n}, \mathcal{T}_n)$ has no CCD. Otherwise we can get a circuit double cover by taking F_n together with the CCD of $(\overline{H_n}, \mathcal{T}_n)$ (after a proper adjustment by adding edges of $H_n \setminus F_n$). The 4-regular graph illustrated in Fig. 3-(a) is the contracted graph $\overline{H_0}$ where the 2-factor F_0 is a pair of edge-disjoint hamiltonian circuits (illustrated by thin lines and thick lines). A study in [11] reveals that each member $(\overline{H_n}, \mathcal{T}_n)$ in this family contains a UD-K_5-minor due to the structure of $(\overline{H_n}, \mathcal{T}_n)$. For example, the resulting transition graph by deleting some edges $\overline{H_0}$ is a subdivision of a

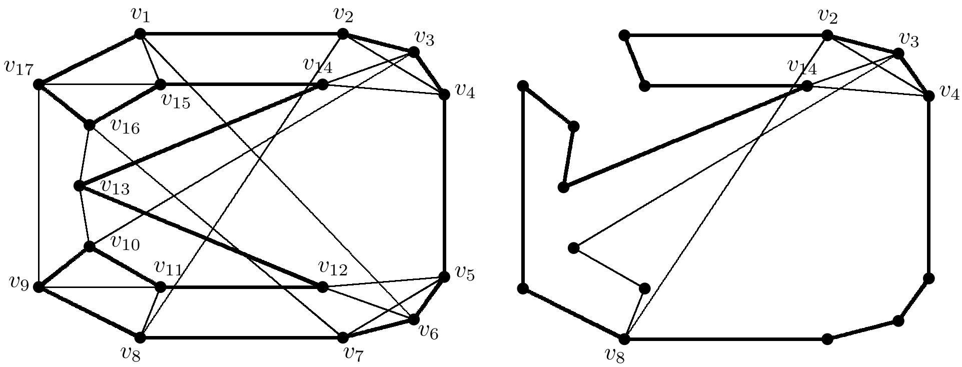

Fig. 3. (a) $(\overline{H_0}, \mathcal{T}_0)$ has no CCD. (b) A UD-K_5-minor in $(\overline{H_0}, \mathcal{T}_0)$.

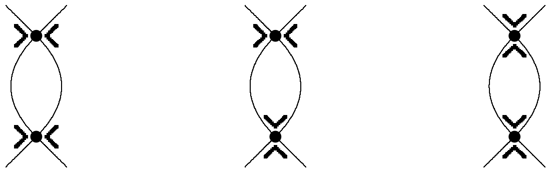

Fig. 4. Digons of type $0, 1$, and 2, respectively.

UD-K_5 (illustrated in Fig. 3-(b)). Therefore, every transitioned 4-regular graph $(\overline{H_n}, \mathcal{T}_n)$ in this family contains a SUD-K_5-minor and does not have a CCD.

2.3. Hamiltonian circuit decomposition, $(X \leftrightarrow O)$-operation, $\langle 2L \rangle$-graphs

Definition 2.13. Let $(G, \mathcal{T})$ be a fully transitioned 4-regular graph. If every CCD of $(G, \mathcal{T})$ is a pair of hamiltonian circuits, then $(G, \mathcal{T})$ is called a Hamilton transitioned graph.

Definition 2.14. Let $D = v_0v_1v_0$ be a digon. D is of type λ where λ is the number of inner vertices of D (see Fig. 4).

Definition 2.15. Let v be a non-trivial degree 4 vertex of a transitioned graph $(G, \mathcal{T})$. The $(X \leftrightarrow O)$-operation at v with $\mathcal{T}(v) = \{\{e_1, e_2\}, \{e_3, e_4\}\}$ is defined as follows (see Fig. 5). Split v with $\{e_1, e_2\}$ becoming incident to a new vertex v_1 and $\{e_3, e_4\}$ incident to another new vertex v_2, and add a pair of parallel edges $\{e_5, e_6\}$ between v_1 and v_2, and define a new transition system by replacing $\mathcal{T}(v)$ with $\mathcal{T}(v_2) = \{\{e_3, e_4\}, \{e_5, e_6\}\}$ and with either $\mathcal{T}(v_1) = \{\{e_1, e_5\}, \{e_2, e_6\}\}$ or $\mathcal{T}(v_1) = \{\{e_1, e_2\}, \{e_5, e_6\}\}$. In fact, we have created a digon of type > 0 between v_1 and v_2.

Definition 2.16. Denote by $\langle 2L \rangle$ the family of all transitioned 4-regular graphs obtained from $(2L, \mathcal{T}_2)$ (which appears on the top left of Fig. 6) by a sequence of $(X \leftrightarrow O)$-operations; it is called the $2L$-family and its members are called $\langle 2L \rangle$-elements.

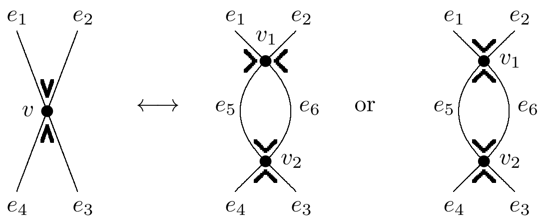

Fig. 5. ($X \leftrightarrow O$)-operations.

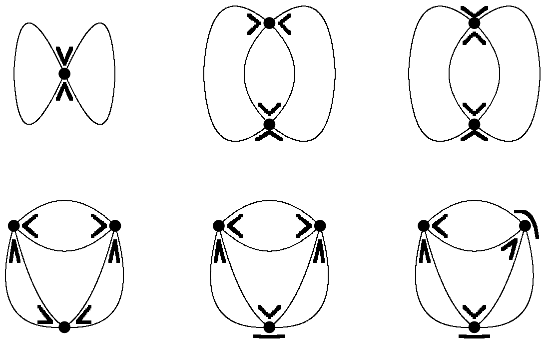

Fig. 6. $\langle 2L\rangle$-elements of order ≤ 3.

Lemma 2.17. *Let $(G,\mathcal{T}) \in \langle 2L\rangle$ be of order at least 3. Then $(G,\mathcal{T})$ has either two vertex-disjoint digons of type ≥ 1, or two edge-disjoint digons of type ≥ 1 with at least one inner transition in the common vertex.*

Proof. Note that the order of $(G,\mathcal{T}) \in \langle 2L\rangle$ being at least 3 implies that G does not contain an edge with multiplicity more than 2 (this is straightforward from the definition of $\langle 2L\rangle$). The family $\langle 2L\rangle$ has precisely three members of order 3 (see Fig. 6); in this case, every $(G,\mathcal{T}) \in \langle 2L\rangle$ has two edge-disjoint digons of type > 0 sharing a common inner vertex.

Thus, the statement of the lemma is true for $(G,\mathcal{T}) \in \langle 2L\rangle$ of order 3. Hence suppose that G is of order greater than 3.

Since ($X \leftrightarrow O$)-operations create a new digon of type > 0, every member of $\langle 2L\rangle$ except $2L$ contains at least one digon of type > 0. Let D be a digon of type $\lambda > 0$ in $(G,\mathcal{T})$ and let $(G',\mathcal{T}') \in \langle 2L\rangle$ be the graph obtained from $(G,\mathcal{T})$ by contracting D. By induction on $|V(G)|$, $(G',\mathcal{T}')$ has either two vertex-disjoint digons of type > 0 or two edge-disjoint digons of type > 0 with an inner transition in a common vertex in each of these two digons. In all cases at least one of these digons of type > 0 and D are either two vertex-disjoint digons of type > 0 or two edge-disjoint digons of type > 0 with inner transitions in the common vertex in $(G,\mathcal{T})$. $\square$

3. Main results

3.1. Compatible circuit decomposition problem and Theorem 1

Given Definition 2.3, Theorem 1 is restated as a stronger version below.

Theorem 1'. *Let $(G, \mathcal{T})$ be an eulerian graph associated with an admissible transition system. If $(G, \mathcal{T})$ is* SUD-K_5*-minor-free, then it has a* CCD*.*

Theorem 1' is not only a graph minor problem, but also a transition minor problem. It was originally proposed by Fleischner [10]. Its weak version for graph minors was solved by Fleischner [7] for planar graphs, and by Fan and Zhang [6] for K_5-minor-free graphs.

Note that Theorem 1' is stronger than the following theorem which is only a graph-minor-free result (not a transition-minor-free result).

Theorem A. [6] *Let $\mathcal{T}$ be an admissible transition system of an eulerian graph G. Then $(G, \mathcal{T})$ has a* CCD *if G is K_5-minor-free.*

3.2. Hamiltonian circuit decomposition problem and Theorem 2

In the studies of circuit covering problems or circuit decomposition problems, one of the critical steps is to determine the structure of the subgraph induced by a pair of incident circuits ([20,21], etc.). The structure of a graph that is covered by or decomposed into a pair of hamiltonian circuits provides a local structure of a possible counterexample to many open problems (such as the circuit double cover conjecture). Its structure for the faithful circuit covering problem was conjectured in [19]; the following is an equivalent version for the corresponding compatible circuit decomposition problem.

Conjecture A. [19] *Let $(G, \mathcal{T})$ be a fully transitioned 4-regular graph such that it has some* CCD *and every such decomposition consists of a pair of hamiltonian circuits. Then $(G, \mathcal{T}) \in \langle 2L \rangle$.*

Theorem 2 solves Conjecture A for SUD-K_5-minor-free graphs. This result generalizes an early result by Lai and Zhang [13] which is a graph minor result for the faithful covering problem.

Note that, in this paper, Theorems 1' and 2 are proved simultaneously, which indicates the technical importance of Hamilton transitioned results (such as, Theorem 2) in the studies of this area.

4. Primary lemmas

4.1. For the proof of Theorem 1'

We consider a counterexample $(G, \mathcal{T})$ to Theorem 1', such that

(1) $|E(G)|$ is as small as possible;
(2) subject to (1), the number of transitions is as small as possible.

$(G, \mathcal{T})$ is called a smallest counterexample to Theorem 1'. It follows from the choice of $(G, \mathcal{T})$ that $(G, \mathcal{T})$ has no removable circuit.

Definition 4.1. Let v be a non-trivial vertex in a transitioned 4-regular graph $(G, \mathcal{T})$. A circuit decomposition of $(G, \mathcal{T})$ is called an almost compatible circuit decomposition with respect to v, if it is compatible in every vertex except v.

A sequence of edge-disjoint circuits $\{C_1, \ldots, C_k\}$ $(k \geq 2)$ of $(G, \mathcal{T})$ is called an almost compatible circuit chain decomposition with respect to v (ACCCD(v) for short), if

(1) it is an almost compatible circuit decomposition with respect to v;
(2) $v \in V(C_1) \cap V(C_k)$, and $v \notin V(C_i)$ $\forall i \in \{2, \ldots, k-1\}$.
(3) for each $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $[V(C_i) \cap V(C_j)] \setminus \{v\} \neq \emptyset$ if and only if $|j-i| = 1$.

The integer k is called the length of the chain $\{C_1, \ldots, C_k\}$ (see Fig. 7).

By an approach similar to the one in [2], [1] and [6], we obtain the following structural results. For the purpose of being self-contained, proofs are therefore included.

Lemma 4.2. [6] *Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1' and let $\mathcal{F}_v = \{C_1, \ldots, C_k\}$ be an* ACCCD *of $(G, \mathcal{T})$ with respect to a non-trivial vertex v. If $k \geq 3$, then $V(C_1) \cap V(C_k) = \{v\}$.*

Proof. By Definition 4.1, $v \in V(C_1) \cap V(C_k)$. Let H be the subgraph induced by $E(C_1) \cup E(C_k)$. If $|V(C_1) \cap V(C_k)| \geq 2$, then $(H, \mathcal{T}|_H)$ is 2-connected. So each C_i, $1 < i < k$, is a removable circuit, which is a contradiction. □

Lemma 4.3. [6] *Any smallest counterexample $(G, \mathcal{T})$ to Theorem 1' is 4-regular, 2-connected, and for every non-trivial vertex v of $(G, \mathcal{T})$, there exists an* ACCCD(v). *Furthermore, every almost CCD with respect to v is an* ACCCD(v).

Proof. Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. Since $\mathcal{T}$ is admissible, $(G, \mathcal{T})$ has no bad cut-vertex. If $\{v\}$ is a 1-separator of G separating G to G_1, G_2, then $(G_1, \mathcal{T}|_{G_1})$ and $(G_2, \mathcal{T}|_{G_2})$ have CCD's $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, Thus, $\mathcal{C}_1 \cup \mathcal{C}_2$ is a CCD of $(G, \mathcal{T})$, a contradiction. Therefore, G is 2-connected.

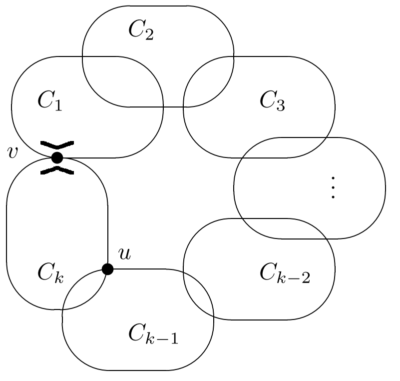

Fig. 7. An ACCCD(v) of $(G, \mathcal{T})$.

Let v be a non-trivial vertex in G and let $(G', \mathcal{T}')$ be a transitioned graph obtained from $(G, \mathcal{T})$ by removing one transition in vertex v, if $\deg(v) > 4$, or by removing all transitions of $\mathcal{T}(v)$, if $\deg(v) = 4$.

By the choice of $(G, \mathcal{T})$, the new graph $(G', \mathcal{T}')$, which has a smaller number of transitions, has a CCD, $\mathcal{F}_v$. Let C_v be the circuit of $\mathcal{F}_v$ containing the vertex v and one of the removed transitions and let $\mathcal{A} = \{C \in \mathcal{F}_v \setminus \{C_v\} | \ C \text{ contains } v.\}$.

By the choice of $(G, \mathcal{T})$, $\mathcal{F}_v$ is an almost compatible circuit decomposition with respect to v.

Construct an auxiliary graph $\mathcal{I}$ with the vertex set $V(\mathcal{I}) = \mathcal{F}_v$ and two vertices of $\mathcal{I}$ are adjacent to each other if and only if their corresponding circuits of $\mathcal{F}_v$ have a non-empty intersection in $G \setminus \{v\}$. Since G is 2-connected, $\mathcal{I}$ is connected. Let $S = C_1 \dots C_k$ be a shortest path in $\mathcal{I}$ from $C_1 = C_v$ to $\mathcal{A}$ ($C_k \in \mathcal{A}$). Obviously, S is a circuit chain of G closed at v.

Let G'' be the subgraph induced by edges of $\cup_{i=1}^{k} E(C_i)$. The transitioned graph $(G'', \mathcal{T}|_{G''})$ is 2-connected, so it has no bad cut-vertex. Thus, every circuit $C \in \mathcal{F}_v \setminus \{C_1, \dots, C_k\}$ is a removable circuit. This is impossible. Therefore, $\mathcal{F}_v = \{C_1, \dots, C_k\}$ is an ACCCD(v) of $(G, \mathcal{T})$ and G is 4-regular. □

Lemma 4.4. *Any smallest counterexample to Theorem 1' has no digon of type $\lambda > 0$.*

Proof. Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. Suppose $(G, \mathcal{T})$ has a digon of type $\lambda > 0$, D. The smaller graph $(G', \mathcal{T}')$ obtained from $(G, \mathcal{T})$ by contracting D remains SUD-K_5-minor-free, because $(G, \mathcal{T})$ has this property. Thus it has a CCD. It is easily seen that every CCD of $(G', \mathcal{T}')$ induces a CCD on $(G, \mathcal{T})$, which is a contradiction. □

Lemma 4.5. *Any smallest counterexample to Theorem 1' is 4-edge-connected.*

Proof. Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. Assume that $\{e_1, e_2\}$ is a 2-edge-cut of $(G, \mathcal{T})$ and G_1, G_2 are the components of $G \setminus \{e_1, e_2\}$. By Lemma 4.3,

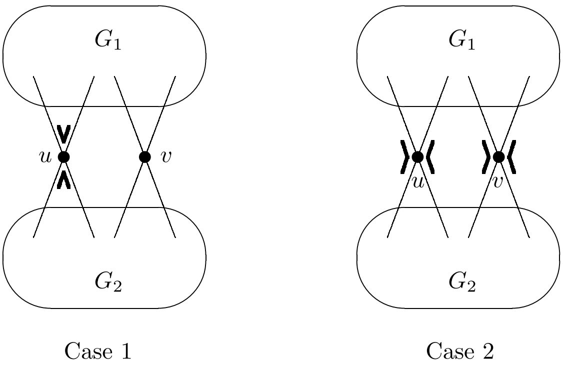

Fig. 8. 2-vertex-cut $\{u, v\}$.

G is 2-connected, so e_1 and e_2 are vertex disjoint. Let $e_1 = u_1u_2$ and $e_2 = v_1v_2$ where $\{u_i, v_i\} \subset V(G_i)$, $i = 1, 2$.

Let $H_i = G/G_{3-i}$ for each $i = 1, 2$. It is easy to check that $(H_i, \mathcal{S}_i)$, $i = 1, 2$, is SUD-K_5-minor-free, $\mathcal{S}_i = \mathcal{T}|_{G_i}$. So there exists a CCD $\mathcal{C}_i$ of $(H_i, \mathcal{S}_i)$ and a circuit $C_i \in \mathcal{C}_i$ covering u_iv_i, $i = 1, 2$. Let $C = (C_1 \cup C_2 \cup \{u_1u_2, v_1v_2\}) \setminus \{u_1v_1, u_2v_2\}$. Thus, $\mathcal{C} = (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C\}) \setminus \{C_1, C_2\}$ is a CCD of $(G, \mathcal{T})$, a contradiction.

Since no eulerian graph has an edge-cut of odd size, $(G, \mathcal{T})$ is 4-edge-connected. □

Lemma 4.6. *Any smallest counterexample to Theorem 1' is 3-connected.*

Proof. Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. By Lemma 4.3, G is a 2-connected 4-regular graph. By Lemma 4.5, $G \setminus X$ has exactly two components, for every 2-vertex-cut X.

Suppose $\{u, v\}$ is a 2-vertex-cut of G such that G_1, G_2 are the components of $G\setminus\{u, v\}$. Every edge-cut in an eulerian graph has an even number of edges. It follows that u, v can be chosen such that for $i = 1, 2$, both u and v have the same degrees in $G \setminus V(G_i)$. By Lemma 4.5, $uv \notin E(G)$ and $\deg_{G\setminus V(G_i)}(u) = \deg_{G\setminus V(G_i)}(v) = 2$, $i = 1, 2$. We have two cases (see Fig. 8).

Case 1. $E(G \setminus V(G_i)) \cap E(u) \in \mathcal{T}(u)$.

In this case, let $(G'_i, \mathcal{T}'_i)$ be a transitioned 4-regular graph obtained from $(G, \mathcal{T})$ by contracting all edges of $G \setminus V(G_i)$. Then, $(G'_i, \mathcal{T}'_i)$ has no SUD-K_5-minor. It follows from the minimality of $(G, \mathcal{T})$ that $(G'_i, \mathcal{T}'_i)$ has a CCD. Then by adapting the circuits containing edges of $E(u) \cup E(v)$ in these two CCD's, we may obtain a CCD of $(G, \mathcal{T})$, which is a contradiction.

Case 2. $\{u_1u, uu_2\} \in \mathcal{T}(u)$, $\{v_1v, vv_2\} \in \mathcal{T}(v)$, where u_i, v_i are neighbours of u and v in G_i, $i = 1, 2$, respectively.

In this case, we set $G'_i = G\setminus V(G_{i+1})$, and define $\mathcal{T}'_i$ as the set of transitions in G'_i induced by $\mathcal{T}|_{G'_i}$. Observe that $(G'_1, \mathcal{T}'_1)$ and $(G'_2, \mathcal{T}'_2)$ have no bad cut-vertex; otherwise, the bad cut-vertex and vertex u is a 2-vertex-cut yielding Case 1.

Therefore, $(G'_i, \mathcal{T}'_i)$ has a CCD, $i = 1, 2$. The union of these two CCD's is a CCD of $(G, \mathcal{T})$, which is a contradiction.

Lemma 4.6 now follows. $\square$

Corollary 4.7. *Any smallest counterexample to Theorem 1' has no digon.*

Proof. Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. Suppose $(G, \mathcal{T})$ has a digon, D. By Lemma 4.4, D is a digon of type 0. Then by Lemma 4.6, $G \setminus E(D)$ is 2-connected. Thus, D is a removable circuit, which is a contradiction. $\square$

Definition 4.8. An even subgraph H of $(G, \mathcal{T})$ is compatible if $|E(H) \cap P| \leq 1$, for every $P \in \mathcal{T}$. An almost compatible 2-even subgraph decomposition $\{U_1, U_2\}$ with respect to v is a decomposition into two even subgraphs in such a way that both U_i's are compatible at every $w \in V(G) \setminus \{v\}$, and U_i is not compatible at v for at least one i.

Definition 4.9. Let $(G, \mathcal{T})$ be a transitioned 4-regular graph. Let v be a non-trivial vertex of degree 4 in $(G, \mathcal{T})$ and let $\{e, f\} \in \mathcal{T}(v)$. By splitting v (with respect to $\mathcal{T}$) we mean that v is split into two degree 2 vertices such that e and f are incident with the same vertex. The split graph of $(G, \mathcal{T})$, denoted by $SP(G, \mathcal{T})$, is the graph obtained from $(G, \mathcal{T})$ by splitting every non-trivial vertex.

The following lemma appeared in [1,6] as part of proofs of some theorems (not as an independent lemma). For the purpose of smoothness of the paper and possible applications in the future, Lemma 4.10 is stated in this paper as an independent lemma. The proof is also included here for the purpose of not only the consistency of notation and terminology but also for the self-completeness of the paper.

Lemma 4.10. [1,6] *Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. Then*

(1) *$SP(G, \mathcal{T})$ has exactly two components;*
(2) *for each non-trivial vertex v, if x and y are the two vertices in $SP(G, \mathcal{T})$ which result by splitting v, then they are contained in different components of $SP(G, \mathcal{T})$;*
(3) *each component of $SP(G, \mathcal{T})$ is a circuit of odd length.*

Proof. Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. By Lemma 4.3, G is 4-regular and for every non-trivial vertex $v \in V(G)$, there exists an ACCCD(v), say $\mathcal{F}_v = \{C_1, \ldots, C_k\}$.

Let

$$S_1 = \cup_{\mu=1}^{\lceil \frac{k}{2} \rceil} E(C_{2\mu-1}) \quad \text{and} \quad S_2 = \cup_{\mu=1}^{\lfloor \frac{k}{2} \rfloor} E(C_{2\mu}).$$

Then, $\{S_1, S_2\}$ is an almost compatible 2-even subgraph decomposition with respect to v. Note that depending on the parity of k, $v \in V(S_2)$ if and only if k is even. If k is odd then S_2 is a set of compatible circuits.

Next, to establish the validity of the Lemma we prove a sequence of claims.

Claim 4.10.1. *For every almost compatible 2-even subgraph decomposition $\{U_1, U_2\}$ with respect to v, for every vertex $w \neq v$, $\deg_{U_i}(w) = 2$, $i = 1, 2$.*

Assume that $\{U_1, U_2\}$ is an almost compatible 2-even subgraph decomposition with respect to v and that there exists a vertex $w \neq v$, $\deg_{U_1}(w) = 4$. By Definition 4.8, a non-trivial vertex of G other than v cannot be of degree 4 in U_i, $i = 1, 2$. Thus, w is a trivial vertex and $E(w) \subseteq E(U_1)$.

Let $\mathcal{F}_i$ be a circuit decomposition of U_i for each $i = 1, 2$. The union $\mathcal{F}_1 \cup \mathcal{F}_2$ forms an almost compatible circuit decomposition with respect to v, by the choice of $(G, \mathcal{T})$. By Lemma 4.3, every almost CCD with respect to a non-trivial vertex is a circuit chain, hence $\mathcal{F}_1 \cup \mathcal{F}_2$ is a circuit chain $\{D_1, \ldots, D_r\}$. Since $G[U_1]$ has a vertex of degree 4, it follows that $r \geq 3$. By Lemma 4.2, we have $V(D_1) \cap V(D_r) = \{v\}$. Let $w \in V(D_j) \cap V(D_{j+1})$. Note that D_j and D_{j+1} are edge-disjoint and both are subsets of U_1. So, every vertex of the induced subgraph $G[D_j \cup D_{j+1}]$ is of degree 2 or 4. If w is the only vertex of $V(D_j) \cap V(D_{j+1})$, then $\{v, w\}$ is a 2-vertex-cut of G (since G has no digon by Corollary 4.7). This contradicts Lemma 4.6.

Thus the induced subgraph $G[D_j \cup D_{j+1}]$ is 2-connected. Let $u_j \in V(D_j) \cap V(D_{j-1})$ (or $u_j = v$ if $j = 1$), and let $u_{j+1} \in V(D_{j+1}) \cap V(D_{j+2})$ (or $u_{j+1} = v$ if $j + 1 = r$). Let $D \subset G[D_j \cup D_{j+1}]$ be a circuit containing the vertices u_j and u_{j+1}. Then $G[D_j \cup D_{j+1}] \backslash D$ is a removable even subgraph of $(G, \mathcal{T})$. This is a contradiction. Thus, $\deg_{U_i}(w) = 2$, for every $w \neq v$, $i = 1, 2$, and thus Claim 4.10.1 is true.

The following claim is obvious.

Claim 4.10.2. *For each circuit C of $SP(G, \mathcal{T})$, $\{S_1 \Delta C, S_2 \Delta C\}$ is also an almost compatible 2-even subgraph decomposition with respect to v.*

Claim 4.10.3. *For each trivial vertex w with $\{e', e''\} = E(w) \cap S_1$, no circuit of $SP(G, \mathcal{T})$ contains both edges e' and e''.*

Suppose that C is a circuit of $SP(G, \mathcal{T})$ containing both edges e' and e''. By Claim 4.10.2, $\{S_1 \Delta C, S_2 \Delta C\}$ is also an almost compatible 2-even subgraph decomposition with respect to v. Note that $\deg_{S_2 \Delta C}(w) = 4$. This contradicts Claim 4.10.1. Thus Claim 4.10.3 now follows.

Therefore, by Claim 4.10.3, we have the following immediate conclusions about $SP(G, \mathcal{T})$.

Let w be a trivial vertex of $(G, \mathcal{T})$.

Claim 4.10.4. *For each pair $\{e', e''\} = E(w) \cap S_i$ $(i = 1, 2)$, the edges e' and e'' must be in different blocks of $SP(G, \mathcal{T})$.*

From Claim 4.10.4, we conclude

Claim 4.10.5. *The trivial vertex w must be a cut-vertex of some component of $SP(G, \mathcal{T})$.*

This also implies

Claim 4.10.6. *The circuit decomposition of $SP(G, \mathcal{T})$ is unique.*

Notation. Let $R_1, \ldots, R_h$ be the components of the split graph $SP(G, \mathcal{T})$, and let $\{X_1, \ldots, X_t\}$ be the unique circuit decomposition of $SP(G, \mathcal{T})$, which is also the block decomposition of $SP(G, \mathcal{T})$.

Claim 4.10.7. *Let x and y be the two vertices in $SP(G, \mathcal{T})$ which result from by splitting v. Then x and y are contained in different components of $SP(G, \mathcal{T})$.*

Proceeding by contradiction, suppose that x and y are contained in the same component R_1, of $SP(G, \mathcal{T})$. Let P be a path of R_1 joining x and y. Let C be the even subgraph induced by $E(P)$ in G. Note that C is not compatible in its vertices except at v. S_1 and S_2 are compatible at every vertex $u \neq v$, and S_1 is not compatible at vertex v. Therefore, $\{S_1 \Delta C, S_2 \Delta C\}$ is a compatible 2-even subgraph decomposition which is a contradiction to the choice of G and thus proves the claim.

By Claim 4.10.7 assume without loss of generality that $x \in X_1$ and $y \in X_2$ where X_j is a block of R_j, $j = 1, 2$.

Claim 4.10.8. *The circuits X_1 and X_2 are of odd lengths, while all other $X_i (i > 2)$ are of even lengths.*

Colour the edges of S_1 with blue, and the edges of S_2 with red. By Claim 4.10.4, each circuit X_i is of even length if $i \neq 1, 2$ since its edges are alternately coloured with red and blue, while X_1 and X_2 are of odd length since each of x, y is incident with two edges of the same colour. Claim 4.10.8 now follows.

The following is the final claim and concludes the proof of the lemma.

Claim 4.10.9. *$h = t = 2$. That is, the split graph $SP(G, \mathcal{T})$ has precisely components $R_1 = X_1$ and $R_2 = X_2$ each of which is a circuit of odd length.*

Since the non-trivial vertex v was selected arbitrarily, all conclusions we have had above can be applied to every non-trivial vertex; that is, for every non-trivial vertex v and the vertices x and y resulting by splitting v, it follows that $x \in X_1$ and $y \in X_2$.

If R_1 has more than one block, then R_1 must have a block Q_3 other than X_1 that contains precisely one cut-vertex z of R_1 (note that Q_3 corresponds to a leaf in the block-cut-vertex graph of R_1). By Claims 4.10.7 and 4.10.8, every vertex of Q_3 is trivial.

So by Claim 4.10.5, every vertex of Q_3 is a cut-vertex of $SP(G,\mathcal{T})$. This contradicts the supposed existence of Q_3.

Furthermore, no edge of R_i with $i > 2$ is incident with a non-trivial vertex. By the definition of $SP(G,\mathcal{T})$, each R_i with $i > 2$ also corresponds to a component of G whose vertices are all trivial. This contradicts G being connected.

Therefore, $SP(G,\mathcal{T})$ consists of two vertex disjoint circuits of odd length $X_1 = R_1$ and $X_2 = R_2$. Lemma 4.10 now follows. □

Since in the proof of Lemma 4.10, it is shown that any smallest counterexample to Theorem 1' has no trivial vertex, we have the following corollary.

Corollary 4.11. *Any smallest counterexample to Theorem 1' is a fully transitioned graph.*

Lemma 4.12. [6] *Let $(G,\mathcal{T})$ be a smallest counterexample to Theorem 1' and let $\mathcal{F}_v = \{C_1,\dots,C_k\}$ be an* ACCCD *of $(G,\mathcal{T})$ with respect to a non-trivial vertex v with $k = |\mathcal{F}_v|$ maximum. Then $k \geq 3$.*

Proof. Since v is of degree 4, $k > 1$ where $\mathcal{F}_v = \{C_1,\dots,C_k\}$. Assume that $k = 2$. Let R_1 and R_2 be the components of $SP(G,\mathcal{T})$ (see Lemma 4.10 (1)). By Lemma 4.10 and Definition 4.9, without loss of generality, let $E(v)\cap E(C_1) \subseteq E(R_1)$ and $E(v) \cap E(C_2) \subseteq E(R_2)$. Consider $\{C_1\Delta R_1, C_2\Delta R_1\}$. It is easy to check that $\{C_1\Delta R_1, C_2\Delta R_1\}$ is an almost compatible decomposition into even subgraphs of $(G,\mathcal{T})$ with respect to v. Note that $E(v) \subseteq E(C_2\Delta R_1)$. Therefore, the maximum degree of $C_2\Delta R_1$ is four and hence any of its circuit decomposition consists of at least two circuits. Since $SP(G,\mathcal{T})$ has two components and G is 2-connected, $(G,\mathcal{T})$ has at least a second non-trivial vertex $u \neq v$. Because C_1 is compatible in u, $C_1\Delta R_1$ is not empty. Therefore, the union of circuit decompositions of $C_1\Delta R_1$ and $C_2\Delta R_1$ has at least three elements. This contradicts the maximality of $|\mathcal{F}_v|$. □

4.2. Cornered triangle extension property: key lemmas for the determination of UD-K_5

There are few results in graph theory that tell us the existence of the Petersen-minor (for example, [5,15], etc.). The main lemmas in this section provide a new approach to identify the precise structure of a transitioned UD-K_5 (their corresponding versions for the faithful circuit covering problem identify the Petersen graph). These lemmas are applied in the final steps of the proofs of Theorems 1' and 2.

Definition 4.13. Let $C_0 = xy_1y_2x$ be a non-compatible circuit of length 3.

(1) The corner of C_0 is a given inner vertex, say x, of the triangle. If y_j is a compatible vertex of C_0, then the opposite edge xy_i is called a leg of C_0 $(i \neq j)$.
(2) For $\mu = 1, 2$, a triangle C_0 with the corner x is called μ-legged if $E(x) \cap E(C_0)$ contains at least μ legs.

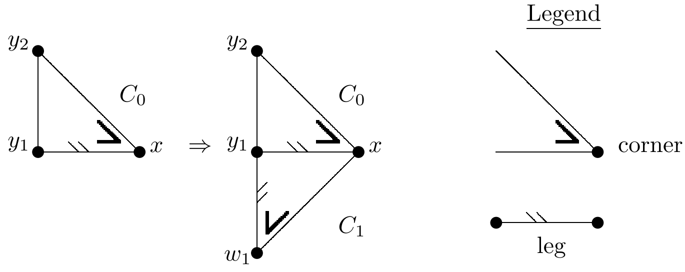

Fig. 9. A cornered triangle $C_0 = xy_1y_2x$, and its extension $C_1 = w_1xy_1w_1$.

(3) Let $C_0 = xy_1y_2x$ be a triangle with the corner x. Given xy_i a leg of C_0, an extension of C_0 along the leg xy_i is another triangle $C_i = w_ixy_iw_i$ with the corner w_i where $w_i \notin V(C_0)$ (note that y_iw_i is a leg of C_i).

(4) A μ-legged triangle $C_0 = xy_1y_2x$ with the corner x is μ-extendable if every leg xy_i has an extension which is also μ-legged (a μ-legged extension; see Fig. 9).

Definition 4.14. For a given integer $\mu \in \{1, 2\}$, a graph G has the the μ-legged-triangle-extension property (abbreviated as μ-LTEP) if G contains some μ-legged triangle and each of them is μ-extendable (see Definition 4.13(4)).

The following two lemmas play an important role in the proofs of the main theorems. These lemmas identify the structure of the UD-K_5 based on the extension property.

In the proofs of the main theorems, the 1-LTEP or 2-LTEP will be verified for smallest counterexamples to the theorems. We wish to point out that although Lemma 4.15 and Lemma 4.16 look very similar, neither of them is an immediate corollary of the other.

Lemma 4.15. *Let $(G, \mathcal{T})$ be a 4-regular, fully transitioned, simple graph. If $(G, \mathcal{T})$ has the* 2-LTEP, *then it is exactly the* UD-K_5.

Proof. By the 2-LTEP, there exists a 2-legged triangle in $(G, \mathcal{T})$, say $S_0 = vv_1v_2v$, with corner v and two legs vv_1 and vv_2. Since S_0 has the 2-LTEP, each leg vv_i $(i = 1, 2)$, has a 2-legged extension $S_i = v_{i+2}vv_iv_{i+2}$ which is also a 2-legged triangle with the corner v_{i+2}.

Since G is simple, it can be seen that $v_3 \neq v_4$, for otherwise, by looking at the transitions contained in $E(v_3)$, the edge vv_3 would be contained in two distinct transitions $\{v_3v, v_3v_1\}$ and $\{v_3v, v_3v_2\}$ (see Fig. 10-(ii)).

Since S_i has the 2-LTEP $(i = 1, 2)$, each leg vv_{i+2} has a 2-legged extension $S_{i+2} = w_ivv_{i+2}w_i$. Since G is 4-regular, $w_1 \in \{v_2, v_4\}$ and $w_2 \in \{v_1, v_3\}$. Since the transition $\{v_4v, v_4v_2\} \in \mathcal{T}(v_4)$ and w_1 is an inner vertex of S_3, we have that $w_1 \neq v_4$. Hence, $w_1 = v_2$. Symmetrically, $w_2 = v_1$.

Since S_1 has the 2-LTEP, the leg v_1v_3, has a 2-legged extension $S_5 = w_3v_1v_3w_3$ with corner w_3. By the 4-regularity of G, $w_3 \in \{v, v_2, v_4\}$. Since w_3 is an inner vertex of S_5,

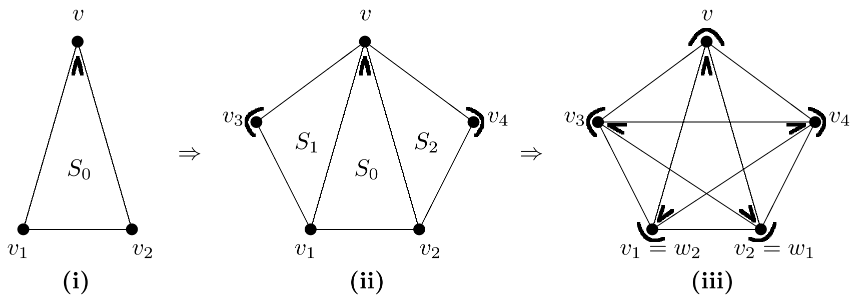

Fig. 10. Proof of Lemma 4.15.

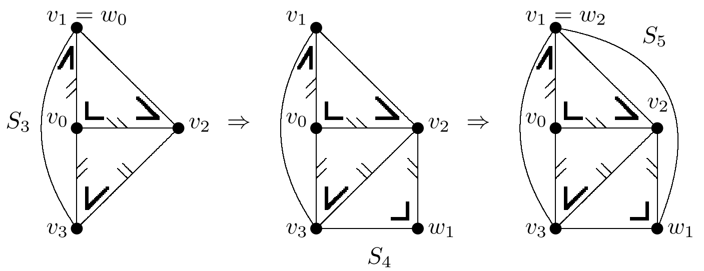

Fig. 11. Case A ($w_0 = v_1$).

one has $w_3 = v_4$ by looking at the transitions at v and v_2. Thus, $\{v_4v_1, v_4v_3\} \in \mathcal{T}(v_4)$, and $\{v_3v_2, v_3v_4\} \in \mathcal{T}(v_3)$ (see Fig. 10-(iii)).

It is now easy to check that $(G, \mathcal{T})$ is exactly the UD-K_5. $\square$

Lemma 4.16. *Let $(G, \mathcal{T})$ be a 4-regular, 4-edge-connected, fully transitioned, simple graph. If $(G, \mathcal{T})$ has the* 1-LTEP*, then either it is the* UD-K_5 *or it has a* CCD *of size* 3.

Proof. Let $S_1 = v_0v_1v_2v_0$ be a 1-legged triangle with the corner v_2 and a leg v_0v_2. By using the 1-LTEP of S_1 at the leg v_0v_2, we have a new vertex v_3 such that $S_2 = v_0v_2v_3v_0$ is a 1-legged triangle with the corner v_3 and a leg v_0v_3.

By using the 1-LTEP of S_2 at the leg v_0v_3, there is a 1-legged triangle $S_3 = v_0v_3w_0v_0$ with the corner w_0 and a leg v_0w_0. Since $S_3 \neq S_2$ and G is simple, there are two possibilities for w_0: $w_0 = v_1$ or $w_0 \notin \{v_0, \ldots, v_3\}$.

Case A: $w_0 = v_1$ (see Fig. 11).

We will show that this case cannot happen.

Since $(G, \mathcal{T})$ is fully transitioned, there exists a transition of v_0 contained in the edge set $\{v_0v_1, v_0v_2, v_0v_3\}$. By rotational symmetry, we may assume that $\{v_0v_1, v_0v_2\} \in \mathcal{T}(v_0)$. Thus v_2v_3 is another leg of the 2-legged triangle S_2. By using the 1-LTEP of S_2 at the leg v_2v_3, there exists a 1-legged triangle $S_4 = v_2v_3w_1v_2$ with the corner w_1 and a leg v_2w_1. It is obvious that $w_1 \notin \{v_0, v_2, v_3\}$. If $w_1 = v_1$, then the edge v_1v_3 will be contained two distinct transitions, which is impossible.

By using the 1-LTEP of S_4 at the leg v_2w_1, there exists a 1-legged triangle $S_5 = v_2w_1w_2v_2$ with the corner w_2 and a leg v_2w_2. Since G is 4-regular and simple, $w_2 \in \{v_0, v_1\}$. If the corner $w_2 = v_0$, then $\{w_2w_1, w_2v_2\} = \{v_0w_1, v_0v_2\} \in \mathcal{T}(v_0)$. But the edge v_0v_2 is already contained in another transition $\{v_0v_1, v_0v_2\}$. This is a contraction, and therefore, $w_2 = v_1$.

Let $e' \in E(v_0) - \{v_0v_1, v_0v_2, v_0v_3\}$ and $e'' \in E(w_1) - \{w_1v_1, w_1v_2, w_1v_3\}$. Since G is 4-regular and 4-edge-connected, we have that $e' = e''$ for otherwise $\{e', e''\}$ is a 2-edge-cut of G. That is, $e' = e'' = w_1v_0$, and $V(G) = \{v_0, v_1, v_2, v_3, w_1\}$.

Consider the 2-legged triangle $v_0w_1v_3v_0$ with corner v_0. By using the 1-LTEP at the leg v_0w_1, there exists a 1-legged triangle $v_0w_1w_3v_0$ with the corner w_3. By the 4-regularity of G, one must have $w_3 = v_1$ or $w_3 = v_2$. However, none of them can happen as can be seen by checking the transitions around v_1 and v_2.

Case B: $w_0 \notin \{v_0, \dots, v_3\}$; denote $w_0 = v_4$ (see Fig. 12).

By using the 1-LTEP of S_3 at the leg v_0v_4, there exists a 1-legged triangle $S_6 = v_0v_4w_3v_0$ with the corner w_3 and a leg v_0w_3. Since G is 4-regular and simple, $w_3 \in \{v_1, v_2\}$. If $w_3 = v_2$, then the edge v_0v_2 is contained in the two transitions $\{v_2v_0, v_2v_1\}$ and $\{v_2v_0, v_2v_4\}$ of v_2. This is a contradiction. Therefore, $w_3 = v_1$.

Note there is no information yet about the transitions around the vertex v_0. By symmetry, there are two cases for further analysis:

$$\{v_0v_1, v_0v_2\} \in \mathcal{T}(v_0) \quad \text{or} \quad \{v_0v_1, v_0v_3\} \in \mathcal{T}(v_0). \tag{1}$$

In either case, we can assume that v_0 is compatible in the triangle $S_2 = v_0v_2v_3v_0$. That is, the edge v_2v_3 is another leg of the triangle S_2. By using the 1-LTEP of S_2 at the leg v_2v_3, we have an extension $S_7 = v_2v_3w_4v_2$ with the corner w_4 and a leg v_2w_4. Proceeding similarly to the above, by looking at the transitions around v_4, we have that $w_4 \neq v_4$. Hence, there are two possibilities for w_4: $w_4 \notin \{v_0, \dots, v_4\}$ or $w_4 = v_1$ (see Fig. 12).

Subcase B-1. $w_4 \notin \{v_0, \dots, v_4\}$; denote $w_4 = v_5$ (see Fig. 13).

For this subcase, we will find a CCD of size 3. By using the 1-LTEP of S_7 at the leg $v_2v_5 = v_2w_4$, there exists an extension $v_2v_5w_5v_2$ with the corner w_5 and a leg v_2w_5. Since G is 4-regular and simple and $w_5 \in [N(v_2) \cap N(v_5)] - V(S_7)$, we have $w_5 = v_1$ (see Fig. 13). Arguing similarly as above, we then get $v_4v_5 \in E(G)$ by the 4-edge connectivity and 4-regularity. Therefore $V(G) = \{v_0, \dots, v_5\}$.

By (1), if $\{v_0v_1, v_0v_3\} \in \mathcal{T}(v_0)$, then consider the 2-legged triangle $S_1 = v_2v_1v_0v_2$ with the corner v_2. The leg v_1v_2 cannot be extended by checking at the transitions around v_5 and the neighbourhood of v_3, v_4. This is a contradiction.

So, by (1), we must have $\{v_0v_1, v_0v_2\} \in \mathcal{T}(v_0)$, and thus the set

$$\{v_1v_2v_3v_4v_1,\ v_0v_1v_5v_3v_0,\ v_0v_2v_5v_4v_0\}$$

is a CCD of $(G, \mathcal{T})$ of size 3.

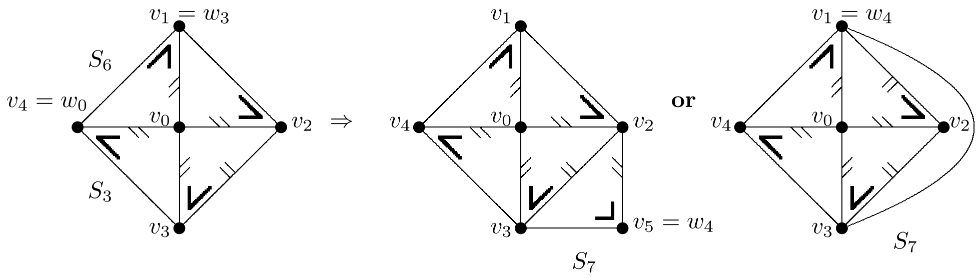

Fig. 12. Case B ($w_0 = v_4$): $S_7 = v_2v_3w_4v_2$ and subcase B-1 ($w_4 = v_5$), subcase B-2 ($w_4 = v_1$).

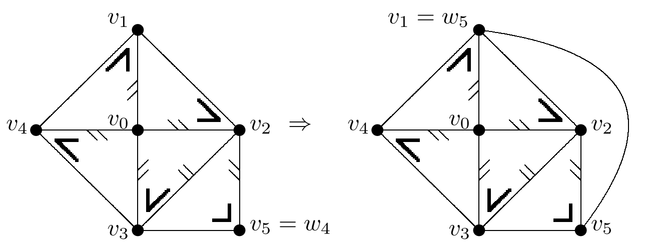

Fig. 13. Subcase B-1 ($w_4 = v_5$).

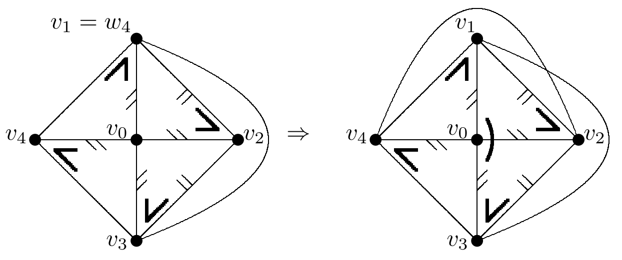

Fig. 14. Subcase B-2 ($v_1 = w_4$): $(G, \mathcal{T})$ is the UD-K_5.

Subcase B-2. $w_4 = v_1$ (see Fig. 14).

It is obvious that $v_2v_4 \in E(G)$ by the 4-edge connectivity and 4-regularity of G (see Fig. 14). By (1), we may first assume that $\{v_0v_1, v_0v_2\} \in \mathcal{T}(v_0)$. Then consider the 2-legged triangle $v_4v_2v_1v_4$ with the corner v_4. The leg v_2v_4 cannot be extended by checking at the transitions around v_0 and v_3. This is a contradiction.

So, by (1), we must have $\{v_0v_1, v_0v_3\} \in \mathcal{T}(v_0)$. It is easy to check that $(G, \mathcal{T})$ is the UD-K_5 (see Fig. 14). $\square$

5. Simultaneous proof of Theorems 1' and 2

Suppose at least one of these two theorems is false. Let $(G, \mathcal{T})$ be a counterexample to either Theorem 1' or Theorem 2 with $|E(G)|$ being as small as possible. Therefore, every admissible transitioned 4-regular graph without SUD-K_5-minor and smaller than $(G, \mathcal{T})$

has a CCD; and for every Hamilton transitioned graph $(H, \mathcal{S})$ smaller than $(G, \mathcal{T})$, if $(H, \mathcal{S})$ is SUD-K_5-minor-free, then $(H, \mathcal{S}) \in \langle 2L \rangle$.

For our considerations we introduce an extra definition.

Definition 5.1. Let G' be a graph obtained from G by some operations. A digon D' of G' is virtual if its corresponding subgraph D in G is a circuit of length > 2 such that at least one edge of D' corresponds to a path of length > 1 in D; otherwise we speak of D' as a real digon.

Now we consider two cases with respect to the assumed counterexample.

Case I. $(G, \mathcal{T})$ is a counterexample to Theorem 1'.
Case II. $(G, \mathcal{T})$ is a counterexample to Theorem 2.

5.1. Case I. $(G, \mathcal{T})$ is a counterexample to Theorem 1'

The goal of our first step is to show that $(G, \mathcal{T})$ has a kind of extension property for a type of cornered triangle, which is to be proved in Lemma 5.5.

Definition 5.2. A circuit $C = v_1 v_2 \dots v_k v_1$ is called an almost removable circuit with respect to v_1 (ARC(v_1), for short) if it is compatible at every vertex except v_1 such that $(G \setminus E(C), \mathcal{T}|_{G \setminus E(C)})$ has no bad cut-vertex.

Note that, for an almost removable circuit C_{v_1} with respect to v_1, if $d(v_1) = 4$ and v_1 is incident with two transitions, say P_1 and P_2, then P_1 is contained in C_{v_1} and P_2 remains in $G \setminus E(C_{v_1})$. If this case happens, the remaining transition P_2 is removed from $T|_{G \setminus E(C_{v_1})}$ by Definition 2.8-(3).

Lemma 5.3. *Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1', and let C_{v_1} be a circuit of G containing v_1. Then C_{v_1} is an* ARC(v_1) *if and only if there exists an* ACCCD(v_1) *$\mathcal{F}_{v_1}$ containing C_{v_1}.*

Proof. Sufficiency is trivially true. Let C_{v_1} be an ARC(v_1). Since $(G, \mathcal{T})$ is a smallest counterexample to Theorem 1', the transitioned graph $(G \setminus E(C_{v_1}), \mathcal{T}|_{G \setminus E(C_{v_1})})$ has a CCD, say $\mathcal{C}_1$. Note that $\mathcal{C}_1 \cup \{C_{v_1}\}$ is an ACCCD(v_1) because of Lemma 4.3. $\square$

Lemma 5.4. *Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1', and let C_{v_1} be a triangle of G containing v_1. If C_{v_1} is compatible at every vertex except v_1, then C_{v_1} is an* ARC(v_1).

Proof. Let $C_{v_1} = v_1 v_2 v_3 v_1$ be compatible at every vertex except v_1. By Definition 5.2, we need to show $(G \setminus E(C_{v_1}), \mathcal{T}|_{G \setminus E(C_{v_1})})$ has no bad cut-vertex. Assume there exists a cut-vertex $x \neq v_1$ in G such that G has two blocks Q_1 and Q_2 incident with x

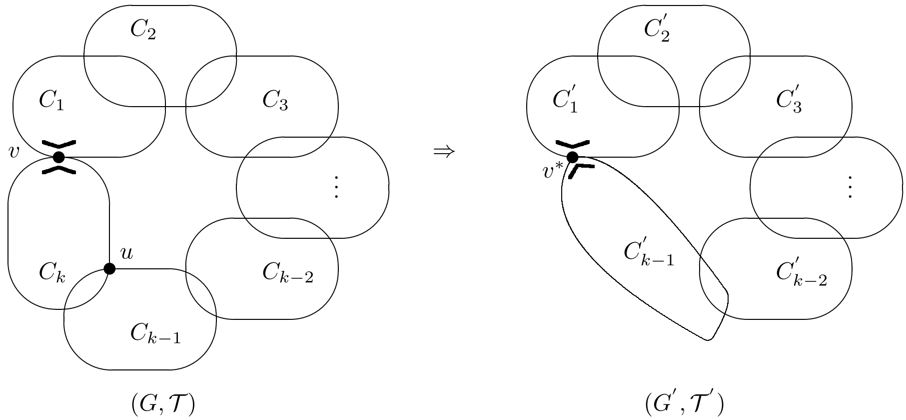

Fig. 15. An ACCCD(v) of $(G, \mathcal{T})$, and, $(G', \mathcal{T}')$.

and $Q_1 \cap E(x) \in \mathcal{T}(x)$. If $V(Q_1) \cap V(C_{v_1}) = \{v_2\}$, then $\{x, v_2\}$ is a 2-vertex-cut. If $V(Q_1) \cap V(C_{v_1}) = \{v_1, v_2\}$, then $\{x, v_3\}$ is a 2-vertex-cut. In both cases we obtain a contradiction to Lemma 4.6. $\square$

Lemma 5.5. *Let $(G, \mathcal{T})$ be a smallest counterexample to Theorem 1'. Then $(G, \mathcal{T})$ has the following properties.*

(i) ARC(v) *exists for every vertex v;*
(ii) *a shortest* ARC *is of length* 3*, and*
(iii) *for every* ARC$(v_1) = v_1v_2v_3v_1$ *and for the edge v_1v_2, there exists an* ARC$(w) = wv_1v_2w$, *$w \neq v_3$.*

Proof. By Lemma 4.3, for every vertex $v \in V(G)$, there exists an ACCCD(v) (see Corollary 4.11), and, for every $v \in V(G)$, by Lemma 5.3, $(G, \mathcal{T})$ contains an ARC(v).

Choose ACR(v) with the smallest length among all ARC's in $(G, \mathcal{T})$ and choose ACCCD(v), $\mathcal{F}_v = \{C_1, \ldots, C_k\}$ with maximum length involving this shortest ACR(v), C_k say (see the left side of Fig. 15).

Let $(G', \mathcal{T}')$ be obtained from $(G, \mathcal{T})$ by deleting all edges of C_k except uv where u is a neighbour of v on C_k, contracting uv to a new vertex v^* and suppressing vertices of degree two.

For every $C' \in G'$, assume that C is the subgraph of $(G, \mathcal{T})$ induced by $E(C')$ and vice versa.

Clearly, $(G', \mathcal{T}')$ has no SUD-K_5-minor (see the right side of Fig. 15), and because of the choice of $(G, \mathcal{T})$, we may consider $\mathcal{F}'$ to be a CCD of $(G', \mathcal{T}')$. There exist two circuits H_1' and H_2' of $\mathcal{F}$ each of which contains the new vertex v^*.

Claim 5.5.1. $\mathcal{F}' = \{H_1', H_2'\}$.

Proof of Claim 5.5.1. Assume that $|\mathcal{F}'| \geq 3$. Then we have to show that, for every $C' \in \mathcal{F}' \setminus \{H_1', H_2'\}$, the corresponding circuit C in G is a removable circuit of $(G, \mathcal{T})$. It is evident that C is compatible in $(G, \mathcal{T})$ since $v^* \notin V(C')$. We thus want to show that $(G \setminus E(C), \mathcal{T}|_{G \setminus E(C)})$ has no bad cut-vertex.

To this end, it is sufficient to show that J is 2-connected where J is the subgraph of G induced by the edges of H_1' and H_2' and the circuit C_k. Note that $H_1' \cup H_2'$ corresponds in G the $H_1 \cup H_2$ which is a pair of paths with the common end-vertices u and v. Adding the circuit C_k, the resulting graph J is therefore 2-connected (because $H_1 \cup H_2 \cup \{uv\}$ is already 2-connected). $\square$

It now follows that every CCD of $(G', \mathcal{T}')$ is a pair of hamiltonian circuits. By the minimality of $(G, \mathcal{T})$, the smaller transitioned graph $(G', \mathcal{T}')$ is not a counterexample to Theorem 2. Thus, we can draw the following conclusion.

Claim 5.5.2.

$$(G', \mathcal{T}') \in \langle 2L \rangle.$$

By Lemma 4.4, $(G, \mathcal{T})$ has no digon of type $\lambda > 0$. However, by Claim 5.5.2 and Lemma 2.17, $(G', \mathcal{T}')$ contains at least two digons of type $\lambda > 0$. Let D' be a digon of type $\lambda > 0$ in $(G', \mathcal{T}')$. Because of Lemma 4.4, there can only be two kinds of digons in $(G', \mathcal{T}')$; either

$$E(D') \cap E(C_{k-1}') \neq \emptyset \neq E(D') \cap E(C_{k-2}')$$

(which is a virtual digon), or D' contains the vertex v^* and some edges of C_1' and C_{k-1}', where $k = 3$ (which is a real digon).

Let D_1' be a virtual digon in $(G', \mathcal{T}')$. Let D_1 denote the circuit in G corresponding to D_1'. Observe that $C_{k-2}' \cap D_1' = C_{k-2} \cap D_1$ is an edge of G and $C_{k-1} \cap D_1$ contains some vertices of C_k. Let $V(D_1') = \{y, z\}$ and let z be an inner vertex of D_1'. If D_1' is of type 2, then it can be easily seen that the circuit $C_{k-1} \Delta D_1$ is a removable circuit in $(G, \mathcal{T})$. Thus, D_1' is of type 1.

Claim 5.5.3. *D_1 is an* ARC(z).

Proof of Claim 5.5.3. Since D_1' is of type 1, it is sufficient to show that $G \setminus E(D_1)$ remains 2-connected.

Suppose $G^* = G \setminus E(D_1)$ has a cut-vertex, x say. Then $x \in V(C_{k-1}) \cap V(C_{k-2})$, since, for every $i \in \{1, \ldots, k\} \setminus \{k-2, k-1\}$, C_i is also as a circuit in G^*. For, if $x \notin V(C_{k-1}) \cap V(C_{k-2})$ would hold, then $\{v, x\}$ would be a 2-vertex-cut in G, contradicting Lemma 4.6. Note that $J = (C_{k-2} \cup C_{k-1}) \setminus E(D_1)$ is a pair of edge-disjoint paths with common end-vertices y and z implying that y and z are not cut-vertices of G^*. Thus, $x \neq y, z$ and x is a cut-vertex of J separating y and z. Let G_1^*, G_2^* be components of $G^* \setminus \{x\}$ with $y \in V(G_1^*)$, $z \in V(G_2^*)$. Let K be the subgraph of G^* induced by the set of circuits $\{C_1, \ldots, C_k\} \setminus \{C_{k-2}, C_{k-1}\}$, which is a connected subgraph of G^* since $v \in V(C_1) \cap V(C_k)$. Then it is easy to see that either $V(K) \subseteq V(G_1^*) \cup \{x\}$ or $V(K) \subseteq V(G_2^*) \cup \{x\}$, but not both. Assume that $V(K) \subseteq V(G_1^*) \cup \{x\}$. Then $\{x, z\}$ is a 2-vertex-cut of G. This contradicts Lemma 4.6 and finishes the proof of the claim. $\square$

By the choice of C_k, the length of D_1 is not smaller than the length of C_k. Thus, by Claim 5.5.3, we have the following immediate corollary.

Claim 5.5.4.

$$V(C_k) \setminus \{v, u\} \subseteq V(C_{k-1}) \cap V(D_1).$$

Claim 5.5.5. $k = 3$.

Proof of Claim 5.5.5. By Lemma 2.17, $(G', \mathcal{T}')$ has at least two edge-disjoint digons of types 1 or 2. If $k \geq 4$, then every digon of $(G', \mathcal{T}')$ is virtual. But, by Claim 5.5.4, at least one of them is a digon of type > 0 in $(G, \mathcal{T})$, contrary to Lemma 4.4. Hence $k = 3$. $\square$

Since $k = 3$, $(G', \mathcal{T}')$ has at most one virtual digon. Let D_2' be a real digon in $(G', \mathcal{T}')$ and let $D_2 = uvwu$ correspond to D_2' in G.

Claim 5.5.6. D_2 *is an* ARC(w) *for some* $w \in V(C_1) \cap V(C_2)$.

Proof of Claim 5.5.6. Denote $D_2' = \langle w, v^* \rangle$ with one edge in C_1' and the other edge in $C_{k-1}' = C_2'$. By the definition of $\mathcal{T}'(v^*)$, D_2' is compatible at v^*. So w is an inner vertex of D_2 since D_2' is of type $\lambda > 0$. D_2' is extended to D_2 in G which is the triangle $vwuv$. If u is also an inner vertex of D_2, then it is easy to see that $C_2 \Delta D_2$ is a removable circuit in $(G, \mathcal{T})$. Now by Lemma 5.4, D_2 is an ARC(w). $\square$

In the general case, by the analogous argument as we did for C_3 and uv, for every ARC(v_1), say $C_{v_1} = v_1v_2v_3v_1$ and the edge v_1v_2, for some $v_1 \in V(G)$, there exists a vertex $w \in (N_G(v_1) \cap N_G(v_2)) \setminus \{v_3\}$ such that $C_w = wv_1v_2w$ is an ARC(w). This completes the proof of the lemma. $\square$

Proof of Theorem 1'. We first claim that every shortest ARC is a 2-legged cornered triangle. Note that, by Definition 5.2, each ARC contains precisely one inner vertex. By Lemma 5.5(ii), every shortest ARC is a triangle. That is, every shortest ARC is a 2-legged cornered triangle.

In order to apply Lemma 4.15, we further claim that $(G, \mathcal{T})$ has the 2-LTEP. By Lemma 5.5(i) and (ii) again, $(G, \mathcal{T})$ contains some 2-legged cornered triangles. By Lemma 5.5(iii), each shortest ARC has an extension at every leg.

Thus, by Lemma 4.15, $(G, \mathcal{T})$ is exactly the UD-K_5, which is a contradiction. $\square$

5.2. Case II. $(G, \mathcal{T})$ is a counterexample to Theorem 2

Lemma 5.6. *$(G, \mathcal{T})$ has no non-hamiltonian removable circuit.*

Proof. Let C be a non-hamiltonian removable circuit of $(G, \mathcal{T})$. Then the SUD-K_5-minor-free transitioned graph $(G \setminus E(C), \mathcal{T}|_{G\setminus E(C)})$ has a CCD $\mathcal{C}$. Thus, $\mathcal{C} \cup \{C\}$ is a CCD of $(G, \mathcal{T})$ with at least three circuits, which is a contradiction. □

Lemma 5.7. *$(G, \mathcal{T})$ has no digon of any type.*

Proof. Suppose that D is a digon of type ≥ 1 in $(G, \mathcal{T})$. Let $(G', \mathcal{T}') = (G/D, \mathcal{T}|_{G/D})$. It is obvious that every CCD of $(G, \mathcal{T})$ induces a CCD on the smaller graph $(G', \mathcal{T}')$ because edges of D of are contained in different members of any CCD. By the same token, every CCD of $(G', \mathcal{T}')$ also induces a CCD of $(G, \mathcal{T})$. Note that $(G', \mathcal{T}')$ remains SUD-K_5-minor-free. Therefore, by the minimality of $(G, \mathcal{T})$, the reduced graph $(G', \mathcal{T}') \in \langle 2L \rangle$. Then, by the definition of the family $\langle 2L \rangle$ of graphs and by the choice of D, we have $(G, \mathcal{T}) \in \langle 2L \rangle$, which is a contradiction.

Assume that $D = \langle v_1, v_2 \rangle$ is a digon of type 0 in $(G, \mathcal{T})$ with $E(D) = \{e_1, e_2\}$. D is a compatible circuit, but not a removable circuit (by Lemma 5.6). Hence, $(G \setminus E(D), \mathcal{T}|_{G\setminus E(D)})$ has a bad cut-vertex w. That is, $\{w\}$ is a 1-separator of $G \setminus E(D)$ separating $G \setminus E(D)$ into two subgraphs G_1 and G_2.

Let $H_i = G/G_j$ for $i \neq j$ and let w_i be the contracted vertex of G_i, for $i = 1, 2$. As an eulerian minor of G, each H_i is SUD-K_5-minor free. And every CCD $\mathcal{F}_i$ of $(H_i, \mathcal{T}|_{H_i})$ has exactly two members for otherwise, a third member of $\mathcal{F}_i$ not passing through the contracted vertex w_i is a removable circuit of $(G, \mathcal{T})$, for $i = 1, 2$. This contradicts Lemma 5.6. Hence, $(G_i, \mathcal{T}|_{H_i})$ remains a Hamilton transitioned graph, and therefore, a member of $\langle 2L \rangle$. By Lemma 2.17, each $(G_i, \mathcal{T}|_{H_i})$ has at least two edge-disjoint digons of type ≥ 1, one of which is different from D and must be a digon of the original graph G. This contradicts the first part of the proof that $(G, \mathcal{T})$ contains no digon of type ≥ 1. □

Definition 5.8. Let $\{H_1, H_2\}$ be a CCD of the Hamilton transitioned graph $(G, \mathcal{T})$. A circuit $C = v_1 v_2 \dots v_k v_1$ is called an H_i-Segment-Chord Circuit with respect to v_1 (H_i-SgCC(v_1) for short) if $v_1 v_k$ is a chord of H_i and $C \setminus \{v_1 v_k\}$ is a segment of H_i and v_1 is an inner vertex of C (See Fig. 16).

Obviously, for every compatible hamiltonian circuit H_i, every transition P at a non-trivial vertex v and every chord e contained in P, there exists an H_i-SgCC(v) containing e.

Lemma 5.9. *For any given decomposition $\{H_1, H_2\}$ into hamiltonian compatible circuits in $(G, \mathcal{T})$ a shortest H_i-SgCC is of length* 3.

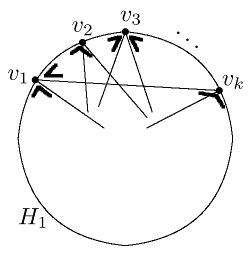

Fig. 16. H_1-SgCC(v_1) $C_0 = v_1v_2 \dots v_kv_1$.

Proof. For $i \in \{1, 2\}$, among all H_i-SgCC's, let $C_0 = v_1 \dots v_kv_1$ be a shortest one. Without loss of generality C_0 is an H_1-SgCC(v_1) (see Fig. 16). By Lemma 5.7, $k \geq 3$.

The new 4-regular graph $(G', \mathcal{T}')$ is obtained from $(G, \mathcal{T})$ by deleting all edges of C_0 except v_1v_k, contracting v_1v_k to a new vertex v^* and suppressing vertices of degree two. $(G', \mathcal{T}')$ remains SUD-K_5-minor-free. Hence, $(G', \mathcal{T}')$ does have a CCD.

Claim 5.9.1. *Every* CCD *of* $(G', \mathcal{T}')$ *is a pair of hamiltonian circuits.*

Let $\mathcal{F}'$ be an arbitrary CCD of $(G', \mathcal{T}')$. There exist two circuits C'_1 and C'_2 in $\mathcal{F}'$ each of which contains the new vertex v^*.

For every circuit $C' \in \mathcal{F}'$, let C denote the subgraph of G induced by the edges of C'. Note that $C_3 = C'_3$ is also a compatible circuit of $(G, \mathcal{T})$, for every circuit $C'_3 \in \mathcal{F}' \setminus \{C'_1, C'_2\}$ if such C'_3 exists. We show that C_3 is removable in $(G, \mathcal{T})$ by showing that the subgraph of G induced by $E(C_0) \cup E(C_1) \cup E(C_2)$ is 2-connected.

Set $H = G[C_1 \cup C_2 \cup (C_0 \setminus \{v_1v_k\})]$; this is the union of three edge-disjoint paths with the common end-vertices v_1 and v_k. If H has a cut-vertex x, it must separate v_1 and v_k. Hence, $H \cup \{v_1v_k\} = C_0 \cup C_1 \cup C_2$ does not have any cut-vertex. Thus, C_3 is a removable circuit of $(G, \mathcal{T})$, for every circuit $C'_3 \in \mathcal{F}' \setminus \{C'_1, C'_2\}$. This contradicts Lemma 5.6. Therefore, $\mathcal{F}' = \{C'_1, C'_2\}$.

Since $(G', \mathcal{T}')$ has no SUD-K_5-minor, by the minimality of $(G, \mathcal{T})$, we draw the following conclusion.

Claim 5.9.2. $(G', \mathcal{T}') \in \langle 2L \rangle$.

Note that v^* is the only contracted vertex of G' and $v_2, \dots, v_{k-1}$ are the only suppressed vertices of G'. Since G has no digon of type $\lambda > 0$ (see Lemma 5.7), for each digon D' of G', the corresponding circuit D of G must contain either some of $\{v_2, \dots, v_{k-1}\}$ or the edge v_1v_k. And if D contains v_1v_k, then D' must contain the contracted vertex v^* and be compatible at v^*.

Claim 5.9.3. *Let D' be a digon of type $\lambda > 0$ in G'. Then the corresponding circuit in G is an H_2-*SgCC*.*

If x is an inner vertex of $D' = \langle x, y\rangle$, then one edge of D' is an H_1-edge, another one is an H_2-segment. So it is an H_2-SgCC(x).

Assume that $k \geq 4$.

Claim 5.9.4. *There is no real digon in G'.*

Suppose to the contrary that there is a real digon D' in G'. Let D be the circuit in G corresponding to D'. Since D is not a digon in G and does not contain any vertex of $\{v_2, \ldots, v_{k-1}\}$, it corresponds to a H_2-SgCC(x) of length 3. This contradicts $k \geq 4$.

Claim 5.9.5. *Every virtual digon uses v^*.*

Let D'_1, D'_2 be a pair of edge-disjoint digons of G'; both are virtual (by Claim 5.9.4).

Suppose that $v^* \notin V(D'_1)$ and x is an inner vertex of D'_1. By Claim 5.9.3, D_1 is an H_2-SgCC(x). By the choice of C_0 (that it is shortest), D_1 must contain all vertices of $\{v_2, \ldots, v_{k-1}\}$. Thus D_2 contains no other suppressed vertices and, therefore, D'_2 is a real digon contradicting Claim 5.9.4.

Claim 5.9.6. *Every virtual digon is compatible at v^*.*

Suppose that v^* is an inner vertex of the digon D'_1. Thus, D_1 is an H_2-SgCC(v_1). We will show that D_1 is shorter than C_0. Since D'_1 and D'_2 are edge-disjoint, each of D'_1, D'_2 contains one transition of $\mathcal{T}'(v^*)$. Hence, v^* must be an inner vertex of both D'_1 and D'_2. Furthermore, the corresponding circuits D_1, D_2 in G do not contain the chord v_1v_k, and contain some vertex of $\{v_2, \ldots, v_{k-1}\}$. That is, D_1 contains at most $(k-3)$ vertices of $\{v_2, \ldots, v_{k-1}\}$. Thus, D_1 is shorter than C_0. This contradicts the choice of C_0.

Claim 5.9.7. $k \leq 4$. *Furthermore, each D_i contains precisely one vertex of $\{v_2, v_3\}$ if $k = 4$.*

Let D'_1, D'_2 be two edge-disjoint digons of G'. Both are virtual, use v^* and are compatible at v^*. And it is obvious that if D'_1 traverses v_n and then D'_2 traverses v_{k+1}. The corresponding circuits D_i in G contain an H_2-segment each passing through at least $k-3$ vertices of $\{v_2, \ldots, v_{k-1}\}$, $i = 1, 2$; for otherwise, it would be shorter than C_0. Since G is 4-regular, $(k-3) + (k-3) \leq k-2$. Thus, $k \leq 4$ and $\{v_2, \ldots, v_{k-1}\} = \{v_2, v_3\}$ implying the validity of the remainder of the claim.

Claim 5.9.8. $k = 3$.

If $k = 4$, then, by Claim 5.9.7, let $D_1 = v_1v_4v_\mu v_nv_1$ with an inner vertex v_n where $\mu = 2$ or 3 (see Fig. 17). Furthermore, the segment $v_4v_\mu v_n$ is an H_2-segment. If $\mu = 2$, then there is a triangle $v_nv_2v_1v_n$ inner at v_n, which is an H_1-SgCC(v_n) shorter than C_0. If $\mu = 3$, then $D^* = \langle v_3, v_4\rangle$ induces a digon of G. This contradicts Lemma 5.7. Thus, $k = 3$ and Lemma 5.9 now follows. □

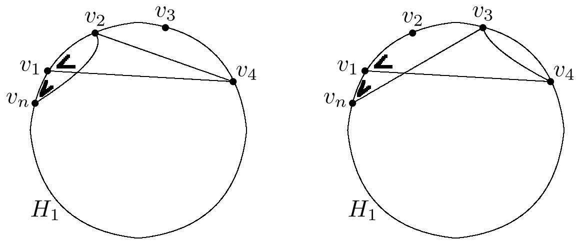

Fig. 17. $k = 4: \;\; D_1 = v_1v_4v_\mu v_nv_1,\ \mu = 2, 3.$

Since $k = 3$ and by Claim 5.9.2, at least one digon of $(G', \mathcal{T}')$ is a real digon, with the circuit corresponding to this digon in $(G, \mathcal{T})$ is a 1-legged triangle $v_1v_3wv_1$ with the corner w and a leg either v_1w or v_3w.

In Lemma 5.9, we proved the existence of 1-legged triangles. In the next lemma (Lemma 5.10), we show that every 1-legged triangle has the 1-LTEP. Note that the proof of this lemma is similar to the proof of Claims 5.9.1 and 5.9.2 for Lemma 5.9.

Lemma 5.10. $(G, \mathcal{T})$ *has the* 1-LTEP.

Proof. Assume that $S_1 = u_1u_2u_3u_1$ is a 1-legged triangle with the corner u_1 and a leg u_1u_3. Let $(G'', \mathcal{T}'')$ be a new 4-regular graph obtaining from $(G, \mathcal{T})$ as follows. Remove u_1u_2 and u_2u_3, contract u_1u_3 to a new vertex u^* and then suppress vertices of degree two. $(G'', \mathcal{T}'')$ remains SUD-K_5-minor-free.

Claim 5.10.1. $(G'', \mathcal{T}'')$ *has no bad cut-vertex.*

Proof of Claim 5.10.1. Suppose that p is a bad cut-vertex in $(G'', \mathcal{T}'')$ ($p \neq u_3$, otherwise u_1 is a cut-vertex of G contrary to G is 2-connected). Thus, $\{u_2, p\}$ is a 2-vertex-cut in $(G, \mathcal{T})$. Let G''_1 and G''_2 be the components of $G \setminus \{u_2, p\}$ such that $\{u_1, u_3\} \subseteq V(G''_1)$.

Remove $V(G''_2)$ and identify u_2 and p to a new vertex q to obtain a new transitioned 4-regular graph $(G''', \mathcal{T}''')$ which is admissible (since $u_1u_3 \in E(G)$,) and SUD-K_5-minor-free. Thus $(G''', \mathcal{T}''')$ has a CCD. It is easily seen that every CCD of $(G''', \mathcal{T}''')$ is a pair of hamiltonian circuits (a removable circuit in $(G''', \mathcal{T}''')$ not containing q is also a removable circuit in $(G, \mathcal{T})$). By the choice of $(G, \mathcal{T})$, $(G''', \mathcal{T}''') \in \langle 2L \rangle$. By Lemma 2.17, $(G''', \mathcal{T}''')$ has two edge-disjoint digons of type > 0. Since $(G, \mathcal{T})$ has no digon of any type, $\{u_1u_2, u_1p\} \in \mathcal{T}(u_1)$. However, $\{u_1u_2, u_1u_3\} \in \mathcal{T}(u_1)$ (see definition of a 1-legged triangle with corner u_1); this contradicts $p \neq u_3$. Now Claim 5.10.1 follows. □

Hence, $(G'', \mathcal{T}'')$ does have a CCD.

Claim 5.10.2. $(G'', \mathcal{T}'') \in \langle 2L \rangle$.

Let $\mathcal{F}''$ be an arbitrary CCD of $(G'', \mathcal{T}'')$. There exist two circuits C''_1 and C''_2 in $\mathcal{F}''$ each of which contains the new vertex u^*.

For every circuit $C'' \in \mathcal{F}''$, denote bz C the subgraph of G induced by the edges of a circuit C''. Note that C_3 is also a compatible circuit of $(G, \mathcal{T})$, for every circuit $C''_3 \in \mathcal{F}'' \setminus \{C''_1, C''_2\}$.

Let H be the subgraph of G induced by the edges contained in C_1, C_2 and $\{u_1u_3\}$, which is the union of three edge-disjoint paths with the common end-vertices u_1 and u_3; and it is 2-connected. Hence, $S_1 \cup C_1 \cup C_2$ is 2-connected. Thus, C_3 is a removable circuit of $(G, \mathcal{T})$, for every circuit $C''_3 \in \mathcal{F}'' \setminus \{C''_1, C''_2\}$ which contradicts Lemma 5.6. Therefore, $\mathcal{F}'' = \{C''_1, C''_2\}$.

Note that $(G'', \mathcal{T}'')$ has no SUD-K_5-minor, thus by the minimality of $(G, \mathcal{T})$, we have $(G'', \mathcal{T}'') \in \langle 2L \rangle$ which finishes the proof of the claim.

By Lemma 2.17, $(G'', \mathcal{T}'')$ has at least two edge-disjoint digons of type $\lambda > 0$. Since $(G, \mathcal{T})$ has no digon by Lemma 5.7, for each digon D'' of $(G'', \mathcal{T}'')$, the corresponding circuit D in G must contain either u_2 or the edge u_1u_3.

There is at most one D in $(G, \mathcal{T})$ with $u_2 \in V(D)$ corresponding to a digon in $(G'', \mathcal{T}'')$; otherwise, $(G, \mathcal{T})$ would contain a digon, contrary to Lemma 5.7. Let $D'' = \langle u^*, w \rangle$ be a digon of type > 0 in $(G'', \mathcal{T}'')$ containing the contracted vertex u^* with edges $\{e_1, e_2\}$ (such digon must exist because of the preceding argument). Because of Lemma 5.7 u^* is not an inner vertex of D''. Its corresponding triangle D in G containing the edge u_1u_3 and therefore $\{e_1, e_2\}$ is not a transition in $\mathcal{T}(u^*)$. Therefore, the only inner vertex of D'' is w. Thus $(G, \mathcal{T})$ has the 1-LTEP. □

Proof of Theorem 2. By Lemma 5.10, $(G, \mathcal{T})$ has the 1-LTEP. Thus by Lemma 4.16, either $(G, \mathcal{T})$ is the UD-K_5 or it has a CCD of size 3, which is a contradiction. Now Theorem 2 follows. □

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