Journal of Combinatorial Theory, Series B 149 (2021) 198–221

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Flows on flow-admissible signed graphs

Matt DeVos [a], Jiaao Li [b,1], You Lu [c,2], Rong Luo [d], Cun-Quan Zhang [d,3], Zhang Zhang [d]

[a] *Department of Mathematics, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6*
[b] *School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China*
[c] *School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China*
[d] *Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA*

ARTICLE INFO

Article history:
Received 20 September 2018
Available online 5 May 2020

Keywords:
Integer flow
Modulo flow
Balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF
Signed graph

ABSTRACT

In 1983, Bouchet proposed a conjecture that every flow-admissible signed graph admits a nowhere-zero 6-flow. Bouchet himself proved that such signed graphs admit nowhere-zero 216-flows and Zýka further proved that such signed graphs admit nowhere-zero 30-flows. In this paper we show that every flow-admissible signed graph admits a nowhere-zero 11-flow.

E-mail address: lijiaao@nankai.edu.cn (J. Li).

[1] This research project has been partially supported by National Natural Science Foundation of China (No. 11901318), Natural Science Foundation of Tianjin (No. 19JCQNJC14100) and the Fundamental Research Funds for the Central Universities, Nankai University (No. 63191425).

[2] This research project has been partially supported by National Natural Science Foundation of China (No. 11871397), the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2020JM-083) and the Fundamental Research Funds for the Central Universities (No. 3102019ghjd003).

[3] This research project has been partially supported by National Science Foundation (DMS-1700218).

https://doi.org/10.1016/j.jctb.2020.04.008
0095-8956/

1. Introduction

Graphs or signed graphs considered in this paper are finite and may have multiple edges or loops. For terminology and notations not defined here we follow [1,4,11].

In 1983, Bouchet [2] proposed a flow conjecture that *every flow-admissible signed graph admits a nowhere-zero* 6*-flow*. Bouchet [2] himself proved that such signed graphs admit nowhere-zero 216-flows; Zýka [13] proved that such signed graphs admit nowhere-zero 30-flows. In this paper, we prove the following result.

Theorem 1.1. *Every flow-admissible signed graph admits a nowhere-zero* 11*-flow.*

In fact, we prove a stronger and very structural result as follows, and Theorem 1.1 is an immediate corollary.

Theorem 1.2. *Every flow-admissible signed graph G admits a* 3*-flow f_1 and a* 5*-flow f_2 such that $f = 3f_1 + f_2$ is a nowhere-zero* 11*-flow, $|f(e)| \neq 9$ for each edge e, and $|f(e)| = 10$ only if $e \in B(supp(f_1)) \cap B(supp(f_2))$, where $B(supp(f_i))$ is the set of all bridges of the subgraph induced by the edges of $supp(f_i)$ $(i = 1, 2)$.*

Theorem 1.2 may suggest an approach to further reduce 11-flows to 9-flows.

The main approach to prove the 11-flow theorem is the following result, which, we believe, will be a powerful tool in the study of integer flows of signed graphs, in particular to resolve Bouchet's 6-flow conjecture.

Theorem 1.3. *Every flow-admissible signed graph admits a balanced nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow.*

A $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow (f_1, f_2) is called *balanced* if $\text{supp}(f_1)$ contains an even number of negative edges.

The rest of the paper is organized as follows: Basic notations and definitions will be introduced in Section 2. Section 3 will discuss the conversion of modulo flows into integer flows. In particular a new result to convert a modulo 3-flow to an integer 5-flow will be introduced and its proof will be presented in Section 5. The proofs of Theorems 1.2 and 1.3 will be presented in Sections 4 and 6, respectively.

2. Signed graphs, switch operations, and flows

Let G be a graph. For $U_1, U_2 \subseteq V(G)$, denote by $\delta_G(U_1, U_2)$ the set of edges with one end in U_1 and the other in U_2. For convenience, we write $\delta_G(U_1)$ and $\delta_G(v)$ for $\delta_G(U_1, V(G) \setminus U_1)$ and $\delta_G(\{v\})$, respectively. The degree of v is the number of edges incident with v, where each loop is counted twice. A d-vertex is a vertex with degree d. Let $V_d(G)$ be the set of d-vertices in G. The maximum degree of G is denoted by $\Delta(G)$. We use $B(G)$ to denote the set of cut-edges of G.

A *signed graph* (G, σ) is a graph G together with a *signature* $\sigma : E(G) \to \{-1, 1\}$. An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* otherwise. Denote the set of all negative edges of (G, σ) by $E_N(G, \sigma)$. For a vertex v in G, we define a new signature σ' by changing $\sigma'(e) = -\sigma(e)$ for each $e \in \delta_G(v)$. We say that σ' is obtained from σ by making a *switch* at the vertex v. Two signatures are said to be *equivalent* if one can be obtained from the other by making a sequence of switch operations. Define the *negativeness* of G by $\epsilon(G, \sigma) = \min\{|E_N(G, \sigma')| : \ \sigma' \text{ is equivalent to } \sigma\}$. A signed graph is balanced if its negativeness is 0. That is it is equivalent to a graph without negative edges. For a subgraph G' of G, denote $\sigma(G') = \prod_{e \in E(G')} \sigma(e)$.

For convenience, the signature σ is usually omitted if no confusion arises or is written as σ_G if it needs to emphasize G. If there is no confusion from the context, we simply use $E_N(G)$ for $E_N(G, \sigma)$ and use $\epsilon(G)$ for $\epsilon(G, \sigma)$.

Every edge of G is composed of two half-edges h and $\hat{h}$, each of which is incident with one end. Denote the set of half-edges of G by $H(G)$ and the set of half-edges incident with v by $H_G(v)$. For a half-edge $h \in H(G)$, we use e_h to refer to the edge containing h. An *orientation* of a signed graph (G, σ) is a mapping $\tau : H(G) \to \{-1, 1\}$ such that $\tau(h)\tau(\hat{h}) = -\sigma(e_h)$ for each $h \in H(G)$. It is convenient to consider τ as an assignment of orientations on $H(G)$. Namely, if $\tau(h) = 1$, h is a half-edge oriented away from its end and otherwise towards its end. Such an ordered triple (G, σ, τ) is called a *bidirected graph.*

Definition 2.1. Assume that G is a signed graph associated with an orientation τ. Let A be an abelian group and $f : E(G) \to A$ be a mapping. The *boundary* of f at a vertex v is defined as

$$\partial f(v) = \sum_{h \in H_G(v)} \tau(h) f(e_h).$$

The pair (τ, f) (or to simplify, f) is an *A-flow* of G if $\partial f(v) = 0$ for each $v \in V(G)$, and is an (integer) *k-flow* if it is a $\mathbb{Z}$-flow and $|f(e)| < k$ for each $e \in E(G)$.

Let f be a flow of a signed graph G. The support of f, denoted by $\text{supp}(f)$, is the set of edges e with $f(e) \neq 0$. The flow f is *nowhere-zero* if $\text{supp}(f) = E(G)$. For convenience, we abbreviate the notions of *nowhere-zero A-flow* and *nowhere-zero k-flow* as *A-NZF* and *k-NZF*, respectively. Observe that G admits an A-NZF (resp., a k-NZF) under an orientation τ if and only if it admits an A-NZF (resp., a k-NZF) under any orientation τ'. A $\mathbb{Z}_k$-flow is also called a modulo k-flow. For an integer flow f of G and a positive integer t, let $E_{f=\pm t} := \{e \in E(G) : |f(e)| = t\}$.

A signed graph G is *flow-admissible* if it admits a k-NZF for some positive integer k. Bouchet [2] characterized all flow-admissible signed graphs as follows.

Proposition 2.2. ([2]) *A connected signed graph G is flow-admissible if and only if $\epsilon(G) \neq 1$ and there is no cut-edge b such that $G - b$ has a balanced component.*

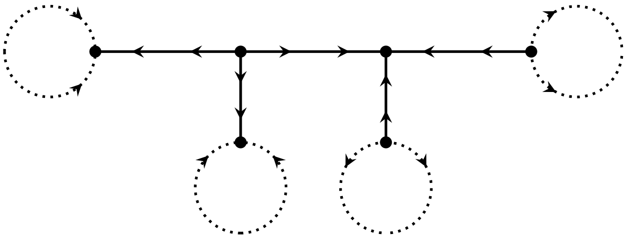

Fig. 1. A signed graph admitting a $\mathbb{Z}_3$-NZF with all edges assigned with 1, but no 4-NZF.

3. Modulo flows on signed graphs

Just like in the study of flows of ordinary graphs and as Theorem 1.3 indicates, the key to make further improvement and to eventually solve Bouchet's 6-flow conjecture is to further study how to convert modulo 2-flows and modulo 3-flows into integer flows. The following lemma converts a modulo 2-flow into an integer 3-flow.

Lemma 3.1 *([3]). If a signed graph is connected and admits a $\mathbb{Z}_2$-flow f_1 such that $supp(f_1)$ contains an even number of negative edges, then it also admits a 3-flow f_2 such that $supp(f_1) \subseteq supp(f_2)$ and $|f_2(e)| = 2$ if and only if $e \in B(supp(f_2))$.*

Remark. In Lemma 3.1 the conclusion "$|f_2(e)| = 2$ if and only if $e \in B(\text{supp}(f_2))$" is not listed in Theorem 1.5 of [3]. However this fact is implicit and follows from the basic property of flows of signed graphs: the flow value of each cut-edge must be even.

In this paper, we will show that one can convert a $\mathbb{Z}_3$-NZF to a very special 5-NZF.

Theorem 3.2. *Let G be a signed graph admitting a $\mathbb{Z}_3$-NZF. Then G admits a 5-NZF g such that $E_{g=\pm 3} = \emptyset$ and $E_{g=\pm 4} \subseteq B(G)$.*

Theorem 3.2 is also a key tool in the proof of the 11-theorem and its proof will be presented in Section 5.

Remark. Theorem 3.2 is sharp in the sense that there is an infinite family of signed graphs that admits a $\mathbb{Z}_3$-NZF but does not admit a 4-NZF. For example, the signed graph obtained from a tree in which each vertex is of degree one or three by adding a negative loop at each vertex of degree one. An illustration is shown in Fig. 1.

4. Proof of the 11-flow theorem

Now we are ready to prove Theorem 1.2, assuming Theorems 1.3 and 3.2.

Proof of Theorem 1.2. Let G be a connected flow-admissible signed graph. By Theorem 1.3, G admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF (g_1, g_2). By Lemma 3.1, G admits a 3-flow f_1 such that $\text{supp}(g_1) \subseteq \text{supp}(f_1)$ and $|f_1(e)| = 2$ if and only if $e \in B(\text{supp}(f_1))$.

By Theorem 3.2, G admits a 5-flow f_2 such that $\text{supp}(f_2) = \text{supp}(g_2)$ and

$$E_{f_2=\pm 3} = \emptyset. \tag{1}$$

Since (g_1, g_2) is a $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF of G,

$$\text{supp}(f_1) \cup \text{supp}(f_2) = \text{supp}(g_1) \cup \text{supp}(g_2) = E(G). \tag{2}$$

We are to show that $f = 3f_1 + f_2$ is a nowhere-zero 11-flow described in the theorem. Since $|f_1(e)| \leq 2$ and $|f_2(e)| \leq 4$, we have

$$|f(e)| = |(3f_1 + f_2)(e)| \leq 3|f_1(e)| + |f_2(e)| \leq 10 \quad \forall e \in E(G).$$

Furthermore, by applying Equations (1) and (2),

$$3f_1(e) + f_2(e) \neq 0, \pm 9 \quad \forall e \in E(G).$$

If $|f(e)| = 10$ for some edge $e \in E(G)$, then $|f_1(e)| = 2$ and $|f_2(e)| = 4$. Thus, by Lemmas 3.1 and 3.2 again, the edge $e \in B(\text{supp}(f_1)) \cap B(\text{supp}(f_2))$ and hence $f = 3f_1 + f_2$ is the 11-NZF described in Theorem 1.2. $\square$

5. Proof of Theorem 3.2

As the preparation of the proof of Theorem 3.2, we first need some necessary lemmas.

The first lemma is a stronger form of the famous Petersen's theorem, and here we omit its proof (see Exercise 16.4.8 in [1]).

Lemma 5.1. *Let G be a bridgeless cubic graph and $e_0 \in E(G)$. Then G has two perfect matchings M_1 and M_2 such that $e_0 \in M_1$ and $e_0 \notin M_2$.*

We also need a splitting lemma due to Fleischner [5].

Let G be a graph and v be a vertex. If $F \subset \delta_G(v)$, we denote by $G_{[v;F]}$ the graph obtained from G by splitting the edges of F away from v. That is, adding a new vertex v^* and changing the common end of edges in F from v to v^*.

Lemma 5.2. ([5]) *Let G be a bridgeless graph and v be a vertex. If $d_G(v) \geq 4$ and $e_0, e_1, e_2 \in \delta_G(v)$ are chosen in a way that e_0 and e_2 are in different blocks when v is a cut-vertex, then either $G_{[v;\{e_0,e_1\}]}$ or $G_{[v;\{e_0,e_2\}]}$ is bridgeless. Furthermore, $G_{[v;\{e_0,e_2\}]}$ is bridgeless if v is a cut-vertex.*

Let G be a signed graph. A path P in G is called a *subdivided edge* of G if every internal vertex of P is a 2-vertex. The *suppressed graph* of G, denoted by $\overline{G}$, is the signed graph obtained from G by replacing each maximal subdivided edge P with a single edge e and assigning $\sigma(e) = \sigma(P)$.

The following result is proved in [12] which gives a sufficient condition when a modulo 3-flow and an integer 3-flow are equivalent for signed graphs.

Lemma 5.3 *([12]). Let G be a bridgeless signed graph. If G admits a $\mathbb{Z}_3$-NZF, then it also admits a 3-NZF.*

Lemma 5.3 is strengthened in the following lemma, which will serve as the induction base in the proof of Theorem 3.2.

Lemma 5.4. *Let G be a bridgeless signed graph admitting a $\mathbb{Z}_3$-NZF. Then for any $e_0 \in E(G)$ and for any $i \in \{1, 2\}$, G admits a 3-NZF such that e_0 has the flow value i.*

Proof. Let G be a counterexample with $\beta(G) := \sum_{v\in V(G)} |d_G(v) - 2.5|$ minimum. Since G admits a $\mathbb{Z}_3$-NZF, there is an orientation τ of G such that for each $v \in V(G)$,

$$\partial\tau(v) := \sum_{h\in H_G(v)} \tau(h) \equiv 0 \pmod 3. \tag{3}$$

We claim $\Delta(G) \le 3$. Suppose to the contrary that G has a vertex v with $d_G(v) \ge 4$. By Lemma 5.2, we can split a pair of edges $\{e_1, e_2\}$ from v such that the new signed graph $G' = G_{[v;\{e_1,e_2\}]}$ is still bridgeless. In G', we consider τ as an orientation on $E(G')$ and denote the common end of e_1 and e_2 by v^*. If $\partial\tau(v^*) = 0$, then $\beta(G') < \beta(G)$ and by Eq. (3), $\partial\tau(u) \equiv 0 \pmod 3$ for each $u \in V(G')$, a contradiction to the minimality of $\beta(G)$. If $\partial\tau(v^*) \ne 0$, then we further add a positive edge vv^* to G' and denote the resulting signed graph by G''. Let τ'' be the orientation of G'' obtained from τ by assigning vv^* with a direction such that $\partial\tau''(v^*) \equiv 0 \pmod 3$. Then by Eq. (3), $\partial\tau''(u) \equiv 0 \pmod 3$ for each $u \in V(G'')$. Since $\beta(G'') < \beta(G)$, we obtain a contradiction to the minimality of $\beta(G)$ again. Therefore $\Delta(G) \le 3$.

Since G is bridgeless, every vertex of G is of degree 2 or 3. Note that the existence of the desired 3-flows is preserved under the suppressing operation. Then the suppressed signed graph $\overline{G}$ of G is also a counterexample, and $\beta(\overline{G}) < \beta(G)$ when G has some 2-vertices. Therefore G is cubic by the minimality of $\beta(G)$.

Since G is cubic, by Eq. (3), either $\partial\tau(v) = d_G(v)$ or $\partial\tau(v) = -d_G(v)$ for each $v \in V(G)$. By Lemma 5.1, we can choose two perfect matchings M_1 and M_2 such that $e_0 \notin M_1$ and $e_0 \in M_2$. For $i = 1, 2$, let τ_i be the orientation of G obtained from τ by reversing the directions of all edges of M_i, and define a mapping $f_i : E(G) \to \{1, 2\}$ by setting $f_i(e) = 2$ if $e \in M_i$ and $f_i(e) = 1$ if $e \notin M_i$. Then f_1 and f_2 are two desired nowhere-zero 3-flows of G under τ_1 and τ_2, respectively, a contradiction. □

Now we are ready to complete the proof of Theorem 3.2.

Proof of Theorem 3.2. We will prove by induction on $t = |B(G)|$, the number of cut-edges in G. If $t = 0$, then G is bridgeless and it is a direct corollary of Lemma 5.4. This establishes the base of the induction.

Assume $t > 0$. Let $e = v_1v_2$ be a cut-edge in $B(G)$ such that one component, say B_1, of $G - e$ is minimal. Let B_2 be the other component of $G - e$. We may assume the bridge e is a positive edge (by possibly some switch operations). Since G admits a $\mathbb{Z}_3$-NZF, $\delta(G) \geq 2$. Thus B_1 is bridgeless and nontrivial. WLOG assume $v_i \in B_i$ $(i = 1, 2)$. Let B'_i be the graph obtained from B_i by adding a negative loop e_i at v_i. Then B'_i admits a $\mathbb{Z}_3$-NZF since G admits a $\mathbb{Z}_3$-NZF. By induction hypothesis, B'_2 admits a 5-NZF g_2 with $g_2(e_2) = a \in \{1, 2\}$. By Lemma 5.4, B'_1 admits a 3-NZF g_1 such that $g_1(e_1) = a$. Hence we can extend g_1 and g_2 to a 5-NZF g of G by setting $g(e) = 2a$ with appropriate orientation of e. Clearly g is a desired 5-NZF of G. □

6. Proof of Theorem 1.3

In this section, we will complete the proof of Theorem 1.3, which is divided into two steps: first to reduce it from general flow-admissible signed graphs to cubic shrubberies (see Lemma 6.6); and then prove that every cubic shrubbery admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF by showing a stronger result (see Lemma 6.13).

We first need some terminology and notations. Let G be a graph. For an edge $e \in E(G)$, *contracting* e is done by deleting e and then (if e is not a loop) identifying its ends. Note that all resulting loops generated from the parallel edges of e are kept. For $S \subseteq E(G)$, we use G/S to denote the resulting graph obtained from G by contracting all edges in S.

For a path P, let $End(P)$ and $Int(P)$ be the sets of the ends and internal vertices of P, respectively. For $U_1, U_2 \subseteq V(G)$, a (U_1, U_2)-path is a path P satisfying $|End(P) \cap U_i| = 1$ and $Int(P) \cap U_i = \emptyset$ for $i = 1, 2$; if G_1 and G_2 are subgraphs of G, we write (G_1, G_2)-path instead of $(V(G_1), V(G_2))$-path. Let $C = v_1 \cdots v_r v_1$ be a circuit. A *segment* of C is the path $v_i v_{i+1} \cdots v_{j-1} v_j \pmod r$ contained in C and is denoted by $v_i C v_j$ or $v_j C^- v_i$. An ℓ-circuit is a circuit with length ℓ.

For a plane graph G embedded in the plane Π, a *face* of G is a connected topological region (an open set) of $\Pi \setminus G$. If the boundary of a face is a circuit of G, it is called a *facial circuit* of G. Denote $[1, k] = \{1, 2, \ldots, k\}$.

6.1. Shrubberies

Now we start to introduce shrubberies and removable circuits, which are key concepts for induction purpose.

Let G be a signed graph and H be a connected signed subgraph of G. An edge $e \in E(G) \setminus E(H)$ is called a *chord* of H if both ends of e are in $V(H)$. We denote the set of chords of H by $\mathcal{C}_G(H)$ or simply $\mathcal{C}(H)$, and partition $\mathcal{C}(H)$ into

$$\mathcal{U}(H) = \mathcal{U}_G(H) = \{e \in \mathcal{C}(H) : H+e \text{ is unbalanced}\} \text{ and } \overline{\mathcal{U}}(H) = \overline{\mathcal{U}_G}(H) = \mathcal{C}(H) \backslash \mathcal{U}(H).$$

A circuit C is called *removable* if either it is unbalanced or it satisfies $|\mathcal{U}(C)| + |V_2(G) \cap V(C)| \geq 2$.

A signed graph G is called a *shrubbery* if it satisfies the following requirements:

(S1) $\Delta(G) \leq 3$;
(S2) every signed cubic subgraph of G is flow-admissible;
(S3) $|\delta_G(V(H))| + \sum_{x \in V(H)}(3 - d_G(x)) + 2|\mathcal{U}(H)| \geq 4$ for any balanced and connected signed subgraph H with $|V(H)| \geq 2$;
(S4) G has no balanced 4-circuits.

The following proposition shows that shrubberies form a nice graph class which is closed under deletion, a crucial fact for induction.

Proposition 6.1. *Every signed subgraph of a shrubbery is still a shrubbery.*

Proof. Let G' be an arbitrary signed subgraph of a shrubbery G. Obviously, G' satisfies (S1), (S2) and (S4). We will show that G' satisfies (S3).

Let H be a balanced and connected signed subgraph of G' with $|V(H)| \geq 2$. Let $A_1 = \delta_G(V(H)) \setminus \delta_{G'}(V(H))$ and $A_2 = \mathcal{C}_G(H) \setminus \mathcal{C}_{G'}(H)$. Then

$$\sum_{x \in V(H)} (3 - d_{G'}(x)) - \sum_{x \in V(H)} (3 - d_G(x)) = \sum_{x \in V(H)} (d_G(x) - d_H(x)) = |A_1| + 2|A_2|.$$

Since $\mathcal{U}_{G'}(H) \subseteq \mathcal{U}_G(H)$ and $\mathcal{C}_{G'}(H) \subseteq \mathcal{C}_G(H)$, we have

$$|\mathcal{U}_G(H)| - |\mathcal{U}_{G'}(H)| \leq |A_2|.$$

Hence

$$\begin{aligned} & |\delta_{G'}(V(H))| + \sum_{x \in V(H)} (3 - d_{G'}(x)) + 2|\mathcal{U}_{G'}(H)| \\ \geq\ & (|\delta_G(V(H))| - |A_1|) + \Big[\sum_{x \in V(H)} (3 - d_G(x)) + |A_1| + 2|A_2| \Big] + 2(|\mathcal{U}_G(H)| - |A_2|) \\ =\ & |\delta_G(V(H))| + \sum_{x \in V(H)} (3 - d_G(x)) + 2|\mathcal{U}_G(H)| \geq 4, \end{aligned}$$

since G is a shrubbery.

Therefore G' satisfies (S3) and thus is a shrubbery. □

Proposition 6.1 will be applied frequently in the proof of Lemma 6.13 and thus it will not be referenced explicitly.

Next we will apply the following two theorems and Lemma 6.5 to reduce Theorem 1.3 for general signed graphs to cubic shrubberies.

Theorem 6.2. ([8]) *Every ordinary bridgeless graph admits a* 6*-NZF.*

Theorem 6.3. ([9]) *Let A be an abelian group of order k. Then an ordinary graph admits a k-NZF if and only if it admits an A-NZF.*

Let G be an ordinary oriented graph, $T \subseteq E(G)$ and A be an abelian group. For any function $\gamma : T \to A$, let $\mathcal{F}_\gamma(G)$ denote the number of A-NZF ϕ of G with $\phi(e) = \gamma(e)$ for every $e \in T$. For every $X \subseteq V(G)$, let $\alpha_X : E(G) \to \{-1, 0, 1\}$ be given by the rule

$$\alpha_X(e) = \begin{cases} 1 & \text{if } e \in \delta_G(X) \text{ is directed toward } X, \\ -1 & \text{if } e \in \delta_G(X) \text{ is directed away } X, \\ 0 & \text{otherwise.} \end{cases}$$

For any two functions γ_1, γ_2 from T to A, we call γ_1, γ_2 *similar* if for every $X \subseteq V(G)$, the following holds

$$\sum_{e\in T} \alpha_X(e)\gamma_1(e) = 0 \text{ if and only if } \sum_{e\in T} \alpha_X(e)\gamma_2(e) = 0.$$

Lemma 6.4. *(Seymour - Personal communication). Let G be an ordinary oriented graph, $T \subseteq E(G)$ and A be an abelian group. If the two functions $\gamma_1, \gamma_2 : T \to A$ are similar, then $\mathcal{F}_{\gamma_1}(G) = \mathcal{F}_{\gamma_2}(G)$.*

Proof. We proceed by induction on the number of edges in $E(G) \setminus T$. If this set is empty, then $\mathcal{F}_{\gamma_i}(G) \le 1$ and $\mathcal{F}_{\gamma_i}(G) = 1$ if and only if γ_i is an A-NZF of G for $i = 1, 2$. Thus, the result follows by the assumption. Otherwise, choose an edge $e \in E(G) \setminus T$. If e is a cut-edge, then $\mathcal{F}_{\gamma_i}(G) = 0$ for $i = 1, 2$. If e is a loop, then we have inductively that

$$\mathcal{F}_{\gamma_1}(G) = (|A| - 1)\mathcal{F}_{\gamma_1}(G - e) = (|A| - 1)\mathcal{F}_{\gamma_2}(G - e) = \mathcal{F}_{\gamma_2}(G).$$

Otherwise, applying induction to $G - e$ and G/e we have

$$\mathcal{F}_{\gamma_1}(G) = \mathcal{F}_{\gamma_1}(G/e) - \mathcal{F}_{\gamma_1}(G - e) = \mathcal{F}_{\gamma_2}(G/e) - \mathcal{F}_{\gamma_2}(G - e) = \mathcal{F}_{\gamma_2}(G). \quad \square$$

The following lemma directly follows from Lemma 6.4.

Lemma 6.5. *Let G be an ordinary oriented graph and A be an abelian group. Assume that G has an A-NZF. If G has a vertex v with $d_G(v) \le 3$ and $\gamma : \delta_G(v) \to A \setminus \{0\}$ satisfies $\partial\gamma(v) = 0$, then there exists an A-NZF ϕ such that $\phi|_{\delta_G(v)} = \gamma$.*

Proof. Let f be an A-NZF of G. Since $d_G(v) \le 3$, $f|_{\delta_G(v)}$ is similar to γ. Thus by Lemma 6.4, we have $\mathcal{F}_\gamma(G) = \mathcal{F}_{f|_{\delta_G(v)}}(G) \neq 0$. Therefore there exists an A-NZF ϕ such that $\phi|_{\delta_G(v)} = \gamma$. □

Now we can reduce Theorem 1.3 to cubic shrubberies.

Lemma 6.6. *The following two statements are equivalent.*

(i) *Every flow-admissible signed graph admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF.*
(ii) *Every cubic shrubbery admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF.*

Proof. "(i)⇒(ii)": By (S2), every cubic shrubbery is flow-admissible, and thus (ii) follows from (i).

"(ii)⇒(i)": Let G be a counterexample to (i) with $\beta(G) = \sum_{v\in V(G)} |d_G(v) - 2.5|$ minimum. Since G is flow-admissible, it admits a k-NZF (τ, f) for some positive integer k and thus $V_1(G) = \emptyset$. Furthermore, by the minimality of $\beta(G)$, G is connected and $V_2(G) = \emptyset$ otherwise the suppressed signed graph $\overline{G}$ of G is also flow-admissible and has smaller $\beta(\overline{G})$ than $\beta(G)$. We are going to show that G is a cubic shrubbery and thus admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF by (ii), which is a contradiction to the fact that G is a counterexample. By the definition of shrubberies, we only need to prove **(I)**-**(III)** in the following.

(I) G is cubic.

Suppose to the contrary that G has a vertex v with $d_G(v) \neq 3$. Then $d_G(v) \ge 4$. Let $\{e_1, e_2\} \subset \delta_G(v)$ and let $G' = G_{[v;\{e_1,e_2\}]}$. Denote the new common end of e_1 and e_2 in G' by v^*. If $\partial f(v^*) = 0$, let $G'' = G'$. If $\partial f(v^*) \neq 0$, we further add a positive edge vv^* with direction from v to v^* and assign vv^* with flow value $\partial f(v^*)$. Let G'' be the resulting signed graph. In both cases, G'' is flow-admissible and $\beta(G'') < \beta(G)$. By the minimality of $\beta(G)$, G'' admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF, and so does G, a contradiction. This proves **(I)**.

(II) $|\delta_G(V(H))| + 2|\mathcal{U}(H)| \ge 4$ for any balanced and connected signed subgraph H with $|V(H)| \ge 2$.

Suppose to the contrary that H is such a subgraph with $|\delta_G(V(H))| + 2|\mathcal{U}(H)| \le 3$. Let $X = V(H)$. Then $H' = G[X] - \mathcal{U}(H)$ is a balanced and connected signed subgraph of G. WLOG assume that all edges of H' are positive. Let $G_1 = G/E(H')$. Then G_1 is also flow-admissible.

Since $|\delta_G(X)| + 2|\mathcal{U}(H)| \le 3$, it follows from the choice of G and Proposition 2.2 that either $|\mathcal{U}(H)| = 0$ and $|\delta_G(X)| \in \{2, 3\}$ or $|\mathcal{U}(H)| = 1$ and $|\delta_G(X)| = 1$. Let x be the contracted vertex in $G_1 = G/E(H')$ corresponding to $E(H')$. Then $d_{G_1}(x) = |\delta_G(X)|+2|\mathcal{U}(H)| \in \{2, 3\}$ and $\beta(G_1) < \beta(G)$ since $|X| = |V(H)| \ge 2$. By the minimality of $\beta(G)$, G_1 admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF (τ_1, f_1), where τ_1 is the restriction of τ on G_1.

Let H_X be the set of the half edges of each edge in $\delta_G(X) \cup \mathcal{U}(H)$ whose end is in X. Then $|H_X| = |\delta_G(X)| + 2|\mathcal{U}(H)| = 2$ or 3. Construct a new graph G_2 from $H' + H_X$ by identifying the non-ends of all half edges in H_X into a new vertex y. Now in G_2, y is the common end of all $h \in H_x$. Then in G_2, y is the vertex incident with all $h \in H_X$. Since G is flow-admissible, G_2 is a bridgeless ordinary graph and thus admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF by Theorems 6.2 and 6.3. Let τ_2 be the restriction of τ on G_2 and define $\gamma(h) = f_1(e_h)$ for each $h \in H_X$. Note that $\tau_2(h) = \tau_1(h)$ for each $h \in H_X$. Since (τ_1, f_1) is a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF of G_1, we have $\partial\gamma(y) = -\partial f_1(x) = 0$. By Lemma 6.5, there is a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF (τ_2, f_2) of G_2 such that $f_2|_{\delta_{G_2}(y)} = \gamma = f_1|_{\delta_{G_1}(x)}$. Thus (τ_1, f_1) can be extended to a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF of G, a contradiction. This proves **(II)**.

(III) G has no balanced 4-circuits.

Suppose to the contrary that G has a balanced 4-circuit C. Then we may assume that all edges of C are positive. Let $G' = G/E(C)$. Then $\beta(G') < \beta(G)$. By the minimality of $\beta(G)$, G' admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF, say (f'_1, f'_2). Since C is a circuit with all positive edges and $|E(C)| = 4$ and since $|\mathbb{Z}_2 \times \mathbb{Z}_3| = 6$, it is easy to extend (f'_1, f'_2) to a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF of G, a contradiction. This proves **(III)** and thus completes the proof of the lemma. □

6.2. Nowhere-zero watering

In this subsection, we will prove that every cubic shrubbery admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF. In fact, we will prove a stronger result that every shrubbery admits a nowhere-zero watering as in Lemma 6.13 below. Here a nowhere-zero watering (see Definition 6.10) involves flows with certain boundaries at vertices of degree one or two, which provides some flexibility for induction and makes some reduction arguments on removable circuits possible. Before proceeding, we need some preparations.

Theorem 6.7. ([10]) *Let G be a 2-connected graph with $\Delta(G) \le 3$ and let $y_1, y_2, y_3 \in V(G)$. Then either there exists a circuit of G containing y_1, y_2, y_3, or there is a partition of $V(G)$ into $\{X_1, X_2, Y_1, Y_2, Y_3\}$ with the following properties:*

(1) $y_i \in Y_i$ *for* $i = 1, 2, 3$*;*
(2) $\delta_G(X_1, X_2) = \delta_G(Y_i, Y_j) = \emptyset$ *for* $1 \le i < j \le 3$*;*
(3) $|\delta_G(X_i, Y_j)| = 1$ *for* $i = 1, 2$ *and* $j = 1, 2, 3$.

Let H be a contraction of G and let $x \in V(G)$. We use $\hat{x}$ to denote the vertex in H which x is contracted into.

Theorem 6.8. ([7]) *Let G be a 2-connected signed graph with $|E_N(G)| = \epsilon(G) = k \ge 2$, where $E_N(G) = \{x_1x_{k+1}, \ldots, x_kx_{2k}\}$. Then the following two statements are equivalent.*

(i) *G does not contain two edge-disjoint unbalanced circuits.*

(ii) *The graph G can be contracted to a cubic graph G' such that either $G' - \{\hat{x}_1\hat{x}_{k+1}, \ldots, \hat{x}_k\hat{x}_{2k}\}$ is a $2k$-circuit C_1 on the vertices $\hat{x}_1, \ldots, \hat{x}_k, \hat{x}_{k+1}, \ldots, \hat{x}_{2k}$ or can be obtained from a 2-connected cubic plane graph by selecting a facial circuit C_2 and inserting the vertices $\hat{x}_1, \ldots, \hat{x}_k, \hat{x}_{k+1}, \ldots, \hat{x}_{2k}$ on the edges of C_2 in such a way that for every pair $\{i, j\} \subseteq [1, k]$, the vertices $\hat{x_i}, \hat{x_j}, \hat{x_{k+i}}, \hat{x_{k+j}}$ are around the circuit C_1 or C_2 in this cyclic order.*

Lemma 6.9. ([6]) *Let G be an ordinary oriented graph and A be an abelian group. Then G is connected if and only if for every function $\beta : V(G) \to A$ satisfying $\sum_{v\in V(G)} \beta(v) = 0$, there exists $\phi : E(G) \to A$ such that $\partial\phi = \beta$.*

Definition 6.10. Let G be a signed graph with $\Delta(G) \leq 3$ and a given orientation. A *nowhere-zero watering* (briefly, NZW) of G is a mapping $f : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_3 - \{(0, 0)\}$ such that

$$\partial f(v) = (0, 0) \text{ if } d_G(v) = 3 \text{ and } \partial f(v) = (0, \pm 1) \text{ if } d_G(v) = 1, 2.$$

Similar to flows, the existence of an NZW is also an invariant under switch operation. The following reductions/extensions of NZW on removable circuits play an important role in later proofs.

Lemma 6.11. *Let G be a shrubbery and C be a removable circuit of G. Then for every NZW $f' = (f_1', f_2')$ of $G' = G - V(C)$, there exists an NZW $f = (f_1, f_2)$ of G so that $f(e) = f'(e)$ for every $e \in E(G')$ and $supp(f_1) = supp(f_1') \cup E(C)$.*

Proof. We first extend f' to $f : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_3$ as follows where α_e is a variable in $\mathbb{Z}_3$ for every $e \in \mathcal{U}(C)$.

$$f(e) = \begin{cases} (0, \pm 1) & \text{if } e \in \delta(V(C)), \\ (1, 0) & \text{if } e \in E(C), \\ (0, 1) & \text{if } e \in \overline{\mathcal{U}}(C), \\ (0, \alpha_e) & \text{if } e \in \mathcal{U}(C). \end{cases}$$

Since every $v \in V(G) \setminus V(C)$ adjacent to a vertex in $V(C)$ has degree less than three in G', we may choose values $f(e)$ for each edge $e \in \delta(V(C))$ so that f satisfies the boundary condition for a watering at every vertex in $V(G) \setminus V(C)$. Obviously by the construction $\partial f_1(v) = 0$ for every $v \in V(C)$. So we need only adjust $\partial f_2(v)$ for $v \in V(C)$ to obtain a watering. We distinguish the following two cases.

Case 1: C is unbalanced.

In this case $\overline{\mathcal{U}}(C) = \emptyset$. Choose arbitrary ± 1 assignments to the variables α_e. Since C is unbalanced, for every vertex $u \in V(C)$, there is a function $\eta_u : E(C) \to \mathbb{Z}_3$ so that $\partial\eta_u(u) = 1$ and $\partial\eta_u(v) = 0$ for any $v \in V(C) \setminus \{u\}$. Now we may adjust f_2 by adding a suitable combination of the η_u functions so that f is an NZW of G, as desired.

Case 2: C is balanced.

WLOG we may assume that every edge of C is positive and every unbalanced chord is oriented so that each half edge is directed away from its end. In this case, each negative chord e contributes $2f_2(e) = \alpha_e$ to the sum $\sum_{v\in V(C)} \partial f_2(v)$. For every $v \in V(C) \cap V_2(G)$, let β_v be a variable in $\mathbb{Z}_3$. Since $|\mathcal{U}(C)| + |V_2(G) \cap V(C)| \geq 2$, we can choose ± 1 assignments to all of the variables α_e and β_v so that the following equation is satisfied:

$$\sum_{v\in V(C)} \partial f_2(v) = \sum_{v\in V(C)\cap V_2(G)} \beta_v.$$

By Lemma 6.9, we may choose a function $\phi : E(C) \to \mathbb{Z}_3$ so that

$$\partial\phi(v) = \begin{cases} \beta_v - \partial f_2(v) & \text{if } v \in V(C) \cap V_2(G), \\ -\partial f_2(v) & \text{if } v \in V(C) \setminus V_2(G). \end{cases}$$

Now modify f by adding ϕ to f_2 and then f is an NZW of G, as desired. $\square$

A *theta* is a graph consisting of two distinct vertices and three internally disjoint paths between them. A theta is *unbalanced* if it contains an unbalanced circuit. By the definition, the following observation is straightforward.

Observation 6.12. *Let G be a signed graph containing no unbalanced thetas and $\Delta(G) \leq 3$. Then for any unbalanced circuit C and any $x \in V(G) \setminus V(C)$, G does not contain two internally disjoint (x, C)-paths.*

Now we present our main result of this subsection.

Lemma 6.13. *Every shrubbery has an NZW. Furthermore, if G is a shrubbery with an unbalanced theta or a negative loop and $\varepsilon \in \{-1, 1\}$, then G has an NZW $f = (f_1, f_2)$ such that $\sigma(supp(f_1)) = \varepsilon$.*

Before we go through the details of the proof, we first present the outline of the proof.

Outline of the proof of Lemma 6.13: Consider G the minimum counterexample to the lemma. If G does not contain an unbalanced theta or a negative loop, by Lemma 6.11, all removable circuits are forbidden from G (See Claim 2-(1)). However due to the requirement of ϵ, if G has an unbalanced theta or a negative loop, only removable circuits with certain properties can be forbidden from G (See Claim 2-(2a) and (2b)).

Thus, in order to avoid "forbidden circuits", certain structures of G are determined step-by-step in Claims 3-8, especially, the non-existence of edge-disjoint unbalanced circuits (Claim 6). With those structures and the application of Theorem 6.8, we are able to lead the final contradiction that some forbidden circuit does exist in the remaining part of the proof (Claims 9-11 and the final step).

Proof. Let G be a minimum counterexample with respect to $|E(G)|$. Then G is connected.

Claim 1. $\Delta(G) = 3$ *and* G *is* 2*-connected. Thus* G *does not contain loops.*

Proof of Claim 1. It is obvious that both a circuit (balanced or unbalanced) and a path have NZWs. Since $\Delta(G) \le 3$ by (S1), we have $\Delta(G) = 3$.

Now we show that G is 2-connected. Suppose to the contrary that G has a cut vertex. Since $\Delta(G) = 3$, G contains a cut-edge $e = v_1v_2$. Let G_i be the component of $G - e$ containing v_i. By the minimality of G, each G_i admits an NZW $f^i = (f_1^i, f_2^i)$, and $\partial f_2^i(v_i) \neq 0$ since $d_{G_i}(v_i) \le 2$. Thus we can obtain an NZW $f = (f_1, f_2)$ of G by setting $f(e) = (0, 1)$ and $f|_{E(G_i)} = f^i$ or $-f^i$ according to the orientation of e and the values of $\partial f_2^1(v_1)$ and $\partial f_2^2(v_2)$. Further, if G contains an unbalanced theta or a negative loop, so does one component of $G - e$, say G_1. By the minimality of G, we choose f^1 such that $\sigma(\text{supp}(f_1^1)) = \epsilon \cdot \sigma(\text{supp}(f_1^2))$. Hence $\sigma(\text{supp}(f_1)) = \sigma(\text{supp}(f_1^1)) \cdot \sigma(\text{supp}(f_1^2)) = \epsilon \cdot \sigma(\text{supp}(f_1^2)) \cdot \sigma(\text{supp}(f_1^2)) = \epsilon$, a contradiction. □

Claim 2. *(1) If* G *does not contain an unbalanced theta, then* G *does not contain a removable circuit.*

(2) If G *contains an unbalanced theta, then* G *has no removable circuit* C *with one of the following properties:*

(2a) $G - V(C)$ *contains an unbalanced theta;*

(2b) $G - V(C)$ *is balanced and* $\sigma(C) = \epsilon$*.*

Proof of Claim 2. Note that G does not contain a negative loop.

(1) is straightforward from Lemma 6.11.

Suppose that (2) is not true. Then G contains an unbalanced theta. Let C be a removable circuit satisfying (2a) or (2b). By the minimality of G, there exists an NZW $f' = (f_1', f_2')$ of $G - V(C)$ such that $\sigma(\text{supp}(f_1')) = \epsilon \cdot \sigma(C)$ in Case (2a) and $\sigma(\text{supp}(f_1')) = 1$ in Case (2b). By Lemma 6.11, f' can be extended to an NZW $f = (f_1, f_2)$ of G such that $\text{supp}(f_1) = \text{supp}(f_1') \cup E(C)$. In particular for Cases (2a) and (2b), $\sigma(\text{supp}(f_1)) = \sigma(\text{supp}(f_1')) \cdot \sigma(C) = \epsilon$, a contradiction. □

Claim 3. *Let* $X \subset V(G)$ *such that* $|X| \ge 2$*,* $G[X]$ *is balanced, and* $|\delta_G(X)| = 2$*. If* $G - X$ *either contains an unbalanced theta, or is balanced and contains a circuit, then* $X \subseteq V_2(G)$ *and thus* $G[X]$ *is a path.*

Proof of Claim 3. The conclusion that $G[X]$ is a path directly follows from the properties of X and the first conclusion that $X \subseteq V_2(G)$.

Suppose the claim fails. Let $X \subset V(G)$ be a minimal set with the above properties such that $X \cap V_3(G) \neq \emptyset$. Then $G[X]$ is 2-connected by the minimality of X. Since $G[X]$

is balanced and $\mathcal{U}(G[X]) = \emptyset$, by (S3), we have

$$2 + \sum_{x\in X}(3 - d_G(x)) = |\delta_G(X)| + \sum_{x\in X}(3 - d_G(x)) + 2|\mathcal{U}(G[X])| \geq 4.$$

The above inequality implies that X contains at least two 2-vertices. Since $G[X]$ is 2-connected, let C be a circuit in $G[X]$ containing at least two 2-vertices. Then C is removable and thus by Claim 2-(2a), $G - V(C)$ does not contain a unbalanced theta, which implies that $G - X$ does not contain unbalanced theta either. By the hypothesis, $G - X$ is balanced and $G - X$ contains a circuit too.

Denote $\delta_G(X) = \{e_1, e_2\}$. Since both $G[X]$ and $G - X$ are balanced, by possibly replacing σ_G with an equivalent signature, we may assume that $\sigma_G(e_1) \in \{-1, 1\}$ and that $\sigma_G(e) = 1$ for every other edge $e \in E(G)$. Since C is a removable circuit of G, G contains an unbalanced theta by Claim 2-(1), and so G is unbalanced. Therefore $\sigma_G(e_1) = -1$ and thus e_1 is the only negative edge in G.

Let C' be an unbalanced circuit and C'' be a circuit in $G - X$. Then C'' is balanced and C' contains e_1 and e_2.

Now we show that $C' \cup (G - X)$ contains an unbalanced theta. Denote $e_1 = x_1y_1$ and $e_2 = x_2y_2$, where $x_1, x_2 \in X$ and $y_1, y_2 \in V(G) \setminus X$. Since G is 2-connected and $\Delta(G) = 3$, there are two disjoint (x_1, C'')-paths P_1 and P_2 with $V(P_1) \cap V(P_2) = \{x_1\}$. Since C' contains both e_1 and e_2, we may choose P_1 and P_2 such that $P_1 \cup P_2$ contains the segment of C' in $G[X]$ from x_1 to x_2. Since e_1 is the only negative edge, $P_1 \cup P_2 \cup C''$ is an unbalanced theta.

Since C' is unbalanced, it is removable. Since $G - V(C')$ is balanced and $\sigma(C') = -1$, by Claim 2-(2b), we have $\epsilon = 1$. On the other hand, since C is removable and $\sigma_G(C) = 1 = \epsilon$, $G - V(C)$ is unbalanced by Claim 2-(2b) again. Thus we may choose the unbalanced circuit C' in $G - V(C)$. Hence $V(C') \cap V(C) = \emptyset$. Therefore $P_1 \cup P_2 \cup C''$ is an unbalanced theta in $G - V(C)$, a contradiction to Claim 2-(2a). □

Claim 4. *Let $X \subset V(G)$ such that $|X| \geq 2$, $G[X]$ is balanced, and $|\delta_G(X)| \leq 3$. For any two distinct ends x_1, x_2 in X of $\delta_G(X)$, there is an (x_1, x_2)-path in $G[X]$ containing at least one vertex in $V_2(G)$.*

Proof of Claim 4. Suppose that the claim fails. Let $x_1x'_1, x_2x'_2 \in \delta_G(X)$, and B_i be the maximal 2-connected subgraph of $G[X]$ containing x_i for $i = 1, 2$. Since G is 2-connected and $\Delta(G) = 3$ by Claim 1 and $|\delta_G(X)| \leq 3$, we have that $G[X]$ is connected and $d_G(x_1) = d_G(x_2) = 3$. Moreover every edge in $\delta_{G[X]}(V(B_i))$ is a cut-edge of $G[X]$ by the maximality of B_i. Thus $|\delta_{G[X]}(V(B_i))|$ is equal to the number of components of $G[X] - V(B_i)$. Since G is 2-connected, we have

(a) for each component A of $G[X] - V(B_i)$, $\delta_G(V(A), V(G) \setminus X) \geq 1$ and thus

(b) $|\delta_G(V(B_i))| \leq |\delta_G(X)| \leq 3$.

Moreover, since $G[X]$ is balanced, B_i is balanced for $i = 1, 2$. Thus we further have

(c) $\mathcal{U}(B_i) = \emptyset$ for $i = 1, 2$.

We first show that for each $i = 1, 2$ B_i does not contain a 2-vertex and is trivial.

WLOG, suppose to the contrary that B_1 contains a 2-vertex y.

If $x_2 \in V(B_1)$, then there are two internally disjoint $(y, \{x_1, x_2\})$-paths P_1 and P_2. Then $P_1 \cup P_2$ is an (x_1, x_2)-path in $G[X]$ containing one 2-vertex.

If $x_2 \notin V(B_1)$, then B_1 and B_2 are disjoint since $\Delta(G) = 3$. Since $G[X]$ is connected, let P_3 be an (x_2, B_1)-path and y_1 be the other end of P_3. Then $y_1 \in V(B_1)$. Again since B_1 is 2-connected and $d_G(x_1) = 3$, $y_1 \neq x_1$ and there are two internally disjoint $(y, \{y_1, x_1\})$-paths, P_1' and P_2'. Then $P_3 \cup P_1' \cup P_2'$ is a desired (x_1, x_2)-path. This proves that B_1 (and B_2) doesn't contain a 2-vertex.

By (b) and (c), we have $|\delta_G(V(B_i))| \leq 3$ and $\mathcal{U}(B_i) = \emptyset$ for $i = 1, 2$. If B_i is nontrivial, then by (S3), we have

$$4 \leq \sum_{x \in V(B_i)} (3 - d_G(x)) + |\delta_G(V(B_1))| \leq \sum_{x \in V(B_i)} (3 - d_G(x)) + 3.$$

The above inequality implies that B_i contains a 2-vertex, a contradiction. Therefore B_i is trivial.

Since $d_G(x_1) = 3$, $d_{G[X]}(x_1) = 2$ and thus $G[X] - x_1$ has two components, say A_1 and A_2. WLOG, we may assume $x_2 \in V(A_2)$. Since G is 2-connected, there exists $x_3x_3' \in \delta_G(V(A_1), V(G) \setminus X)$ with $x_3 \in V(A_1)$. Similarly, $G[X] - x_2$ has two components A_3 and A_4. Since $G[X]$ is connected, the subgraph induced by $V(A_1)$ together with x_1 must be contained in one of A_3 and A_4, say A_4. Thus $\delta_G(V(A_4), V(G) \setminus X) = \{x_1x_1', x_3x_3'\}$. Note that $\delta_G(X) = \{x_1x_1', x_2x_2', x_3x_3'\}$ since $|\delta_G(X)| \leq 3$. Since $x_2 \notin V(A_3)$, $\delta(V(A_3), V(G) \setminus X) = 0 < 1$, a contradiction to (a). This proves the claim. □

Claim 5. *G does not contain two disjoint unbalanced circuits C_1 and C_2 such that $V_3(G) \subseteq V(C_1) \cup V(C_2)$.*

Proof of Claim 5. Suppose the claim fails. Let C_1 and C_2 be two disjoint unbalanced circuits such that $V_3(G) \subseteq V(C_1) \cup V(C_2)$. Then every vertex of $G' = G - E(C_1 \cup C_2)$ is of degree at most 2. By Claim 2-(2a), $G - V(C_i)$ does not contain unbalanced theta for each $i = 1, 2$. Thus by Observation 6.12, every nontrivial component of G' is a path with one end in $V(C_1)$ and the other end in $V(C_2)$. Since G is 2-connected and $\Delta(G) = 3$, there are at least two 3-vertices in each C_i.

When $\epsilon = -1$, choose x_1, x_2 from $V_3(G) \cap V(C_1)$ such that the segment $P = x_1C_1x_2$ contains all vertices of $V_3(G) \cap V(C_1)$. Let P_i be the path in G' with one end x_i and y_i be the other end of P_i for $i = 1, 2$. Since C_2 is unbalanced, there is a segment, say $y_1C_2y_2$, of C_2 such that the circuit $C = P \cup P_1 \cup P_2 \cup y_1C_2y_2$ is unbalanced, and thus C is removable. This contradicts Claim 2-(2b) since $G - V(C)$ is a forest (which is balanced).

When $\epsilon = 1$, by the minimality of G and since $G'' = G - V(C_1 \cup C_2)$ is a forest, G'' admits an NZW $f' = (f_1', f_2')$ with $\text{supp}(f_1') = \emptyset$. By applying Lemma 6.11 twice, we extend $f' = (f_1', f_2')$ to an NZW $f = (f_1, f_2)$ of G such that $\text{supp}(f_1) = E(C_1) \cup E(C_2)$. So $\sigma(\text{supp}(f_1)) = \sigma(C_1) \cdot \sigma(C_2) = 1 = \epsilon$, a contradiction. □

Claim 6. *G does not contain two disjoint unbalanced circuits.*

Proof of Claim 6. Suppose to the contrary that C_1 and C_2 are two disjoint unbalanced circuits of G. By Claim 5, $V_3(G) \setminus V(C_1 \cup C_2) \neq \emptyset$.

Let $x \in V_3(G) \setminus V(C_1 \cup C_2)$. By Claim 2-(2a), for each C_i, $G - V(C_i)$ does not contain an unbalanced theta. Thus by Observation 6.12, there exists a 2-edge-cut of G separating x from $V(C_1 \cup C_2)$. Let $\{e_1, e_2\}$ be such a 2-edge-cut. Let

$$\mathcal{F} = \{e_1\} \cup \{e \in E(G) : \{e, e_1\} \text{ is a 2-edge-cut of } G\}$$

and $\mathcal{B}$ be the set of all nontrivial components of $G - \mathcal{F}$. Then every member of $\mathcal{B}$ is 2-connected. Since $d_G(x) = 3$, there is a $B_0 \in \mathcal{B}$ containing x.

We claim that $\mathcal{B}$ has the following properties:
(a) Each $B \in \mathcal{B}$ contains a removable circuit. In particular, if B is balanced, then B contains at least one 2-vertex.
(b) Each $B \in \mathcal{B}$ is either balanced or is an unbalanced circuit.
(c) $|\mathcal{B}| \geq 3$.

Let $B \in \mathcal{B}$. Then $|\delta_G(V(B))| = 2$ and $\mathcal{U}(B) = \emptyset$. If B is balanced, then by (S3), B contains at least two 2-vertices and thus contains a circuit containing at least two 2-vertices which is removable. If B is unbalanced, then B contains an unbalanced circuit which is also removable. This proves (a).

Since B_0 doesn't contain C_1 or C_2, $|\mathcal{B}| \geq 2$. By (a) each member B in $\mathcal{B}$ contains a removable circuit. Thus by Claim 2-(2a), each member of $\mathcal{B}$ does not contain unbalanced theta and so is an unbalanced circuit if it is unbalanced. This proves (b).

By (b), C_1 and C_2 belong to distinct members in $\mathcal{B}$. Note that B_0 doesn't contain C_1 or C_2. Thus $|\mathcal{B}| \geq 3$. This proves (c).

Since G is 2-connected, there is a circuit that contains all edges in $\mathcal{F}$ and goes through every $B \in \mathcal{B}$. We choose such a circuit C with the following properties:
(1) $\sigma(C) = \epsilon$ (the existence of C is guaranteed since C_1 is unbalanced);
(2) subject to (1), $|V_2(G) \cap V(C - V(C_1))|$ is as large as possible;
(3) subject to (1) and (2), $|E_N(G) \cap E(C - V(C_1))|$ is as small as possible.

We claim that C is removable.

Let $B \in \mathcal{B} \setminus \{C_1\}$. If B is balanced, then by (a), B contains a 2-vertex. Since B is 2-connected, by (2), C contains at least one 2-vertex in B. If B is an unbalanced circuit of length at least 3, then by (2), C contains one 2-vertex in B too. If B is an unbalanced circuit of length 2, then by (3), C contains the positive edge in B and the negative edge in B belongs to $\mathcal{U}(C)$. Therefore every $B \in \mathcal{B} \setminus \{C_1\}$ contributes at least 1 to $|\mathcal{U}(C)| + |V_2(G) \cap V(C)|$. Since $|\mathcal{B} \setminus \{C_1\}| \geq 2$, we have $|\mathcal{U}(C)| + |V_2(G) \cap V(C)| \geq 2$. Hence C is a removable circuit.

Since each $B \in \mathcal{B}$ is either balanced or an unbalanced circuit, $G - V(C)$ is balanced. This contradicts Claim 2-(2b) since C is removable and since $\sigma(C) = \epsilon$ by (1). □

Claim 7. *G contains an unbalanced theta and $\epsilon = 1$.*

Proof of Claim 7. We first show that G contains an unbalanced theta.

Suppose that G does not contain unbalanced theta. If G is unbalanced, then it contains an unbalanced circuit. If G is balanced, then $|V_2(G)| = \sum_{x\in V(G)}(3 - d_G(x)) \geq 4 - |\delta_G(V(G))| - |\mathcal{U}(G)| = 4$ by (S3). Since G is 2-connected by Claim 1, G has a circuit containing at least two 2-vertices. Hence G has a removable circuit in either case. It contradicts Claim 2-(1). Therefore G contains an unbalanced theta.

The existence of unbalanced thetas implies that $\epsilon \in \{-1, 1\}$. Let C be an unbalanced circuit. By Claim 6, G does not contain two disjoint unbalanced circuits, and thus $G - V(C)$ is balanced. By Claim 2-(2b), $\epsilon \neq \sigma(C) = -1$, so $\epsilon = 1$. □

Claim 8. $|E_N(G)| \geq 2$.

Proof of Claim 8. By Claim 7, G is unbalanced. Suppose to the contrary that $E_N(G) = \{e_0\}$. Let P be the maximal subdivided edge of G containing e_0. Let y_0, y_1 be the two ends of P. Then $Int(P) \subseteq V_2(G)$ and $y_0, y_1 \in V_3(G)$. Let $G' = G - Int(P)$ if $Int(P) \neq \emptyset$; Otherwise, let $G' = G - e_0$.

We claim that G' is 2-connected. Suppose to the contrary that G' is not 2-connected. Let B be the maximal 2-connected subgraph of G' containing y_1. Since $G = G' \cup P$ is 2-connected by Claim 1, $y_0 \notin V(B)$ and $\delta_{G'}(V(B)) \neq \emptyset$. By the maximality of B, each edge in $\delta_{G'}(V(B))$ is a cut-edge of G'. Since G is 2-connected again, $|\delta_{G'}(V(B))| = 1$ and thus $|\delta_G(V(B))| = 2$ and B is nontrivial since $d_G(y_1) = 3$. Similarly the maximal 2-connected subgraph of G' containing y_0 is nontrivial and thus contains a circuit. Therefore B is balanced and $G - V(B)$ is balanced and contains circuits since $E_N(G) = \{e_0\} \subseteq E(P)$. By Claim 3, $V(B) \subseteq V_2(G)$, which contradicts the fact $y_1 \in V_3(G)$. This proves that G' is 2-connected.

(i) G' does not contain a circuit C such that $\{y_0, y_1\} \cap V(C) \neq \emptyset$ and $|V(C) \cap V_2(G)| \geq 2$.

Proof of *(i)*. Otherwise, C is a removable circuit such that $G - V(C)$ is balanced and $\sigma(C) = 1 = \epsilon$ by Claim 7, a contradiction to Claim 2-(2b).

Since G' is balanced and 2-connected, and is also a shrubbery by Proposition 6.1, $|V_2(G')| = \sum_{x\in V(G')}(3 - d_{G'}(x)) \geq 4$ by (S3) and thus at least two vertices in $V_2(G')$, say y_2 and y_3, also belong to $V_2(G)$. Note that $\{y_2, y_3\} \cap \{y_0, y_1\} = \emptyset$. By (*i*), there is no circuit in G' containing $\{y_1, y_2, y_3\}$. Thus by Theorem 6.7, there is a partition of $V(G')$ into $\mathcal{I} = \{X_1, X_2, Y_1, Y_2, Y_3\}$ such that $y_i \in Y_i$ $(i = 1, 2, 3)$, $\delta_{G'}(X_1, X_2) = \delta_{G'}(Y_i, Y_j) = \emptyset$ $(1 \leq i < j \leq 3)$, and $\delta_{G'}(X_i, Y_j) = e_{ij}$ $(i = 1, 2;\ j = 1, 2, 3)$. See Fig. 2. For each $Z \in \mathcal{I}$, $G'[Z]$ is connected since G' is 2-connected and $|\delta_{G'}(Z)| \leq 3$.

Since G' is 2-connected and $|\delta_{G'}(Y_j)| = 2$ for $j \in \{2, 3\}$, we have the following statement.

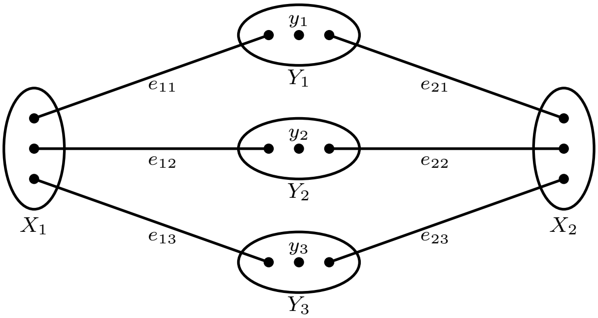

Fig. 2. A partition of $V(G')$ into $\mathcal{I} = \{X_1, X_2, Y_1, Y_2, Y_3\}$.

(ii) For any $\{i,j\} = \{2,3\}$, there is a circuit C_i in $G' - Y_j$ containing y_1 and all the edges in $\{e_{11}, e_{1i}, e_{2i}, e_{21}\}$. We choose C_i such that $|V(C_i) \cap V_2(G)|$ is as large as possible. Then by (i), $|V(C_i) \cap V_2(G)| \leq 1$.

(iii) $y_0 \notin Y_2 \cup Y_3$, $Y_2 = \{y_2\}$, and $Y_3 = \{y_3\}$.

Proof of *(iii)*. Let $j \in \{2,3\}$. We first show $|Y_j| = 1$ if $y_0 \notin Y_j$. WLOG suppose to the contrary $y_0 \notin Y_3$ and $|Y_3| \geq 2$. Since $G = G' \cup P$ and $y_0 \notin Y_3$, $|\delta_G(Y_3)| = |\delta_{G'}(Y_3)| = 2$. By (ii), C_2 is a circuit in $G' - Y_3$. Since $G'[Z]$ is connected for each $Z \in \mathcal{I}$, $G' - Y_3$ is connected. Thus there is a (y_0, C_2)-path P' in $G' - Y_3$, so $P' \cup P \cup C_2$ is an unbalanced theta in $G - Y_3$. Since $G[Y_3]$ is balanced and $|\delta_G(Y_3)| = 2$, by Claim 3, $Y_3 \subseteq V_2(G)$ and $G[Y_3]$ is a path. Thus $Y_3 \subset V(C_3)$ and $|V(C_3) \cap V_2(G)| \geq 2$, a contradiction to (ii). This proves $|Y_3| = 1$. Therefore $|Y_j| = 1$ if $y_0 \notin Y_j$ for each $j \in \{2,3\}$.

Now we show $y_0 \notin Y_2 \cup Y_3$. Otherwise WLOG, assume $y_0 \notin Y_3$ and $y_0 \in Y_2$. Then $Y_3 = \{y_3\}$ and $y_3 \in V_2(G)$. By (S4), C_3 is not a balanced 4-circuit, and thus there is a set $Z \in \{Y_1, X_1, X_2\}$ such that $|V(C_3) \cap Z| \geq 2$. Since $|V(Z) \cap \{y_0, y_1\}| \leq 1$, $G[Z]$ is balanced. Obviously $|\delta_G(Z)| = 3$. By Claim 4 and the maximality of $|V(C_3) \cap V_2(G)|$, C_3 contains a 2-vertex in Z. Together with the 2-vertex y_3, we have $|V(C_3) \cap V_2(G)| \geq 2$, a contradiction to (ii). This shows $y_0 \notin Y_2 \cup Y_3$ and thus $|Y_2| = |Y_3| = 1$.

(iv) $|X_i| = 1$ if $y_0 \notin X_i$ for any $i \in \{1,2\}$ and thus $y_0 \in X_1 \cup X_2$.

Proof of *(iv)*. Suppose that for some $i \in \{1,2\}$, $y_0 \notin X_i$ and $|X_i| \geq 2$. WLOG assume $i = 1$. Let x_{1j} be the end of e_{1j} in X_1 for $j = 1,2,3$. Since $|X_1| \geq 2$ and since $\Delta(G) = 3$ and G is connected by Claim 1, $x_{11} \neq x_{1j}$ for some $j \in \{2,3\}$. Note that $x_{11}, x_{1j} \in V(C_j)$. Since $|\delta_G(X_1)| = 3$ and $G[X_1]$ is balanced, by Claim 4, there is an (x_{11}, x_{1j})-path in X_1 containing a 2-vertex. So C_j contains a 2-vertex in X_1 by the maximality of $|V(C_j) \cap V_2(G)|$. Since $d_G(y_j) = 2$ and C_j contains y_j, $V(C_3)$ contains at least two 2-vertices, a contradiction to (ii). This proves that $|X_i| = 1$ if $y_0 \notin X_i$ for any $i \in \{1,2\}$.

If $y_0 \notin X_1 \cup X_2$, then $|X_1| = |X_2| = 1$. By (iii), $G[Y_2 \cup Y_3 \cup X_1 \cup X_2]$ is a balanced 4-circuit, a contradiction to (S4). Therefore $y_0 \in X_1 \cup X_2$.

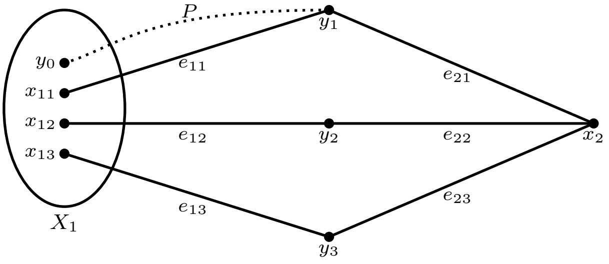

Fig. 3. $G' = G - Int(P) - E(P)$.

By (iv), WLOG assume $y_0 \in X_1$. Then by (iv) and (iii), $|X_2| = |Y_2| = |Y_3| = 1$. Denote $X_2 = \{x_2\}$.

(v) $Y_1 = \{y_1\}$.

Proof of *(v)*. Suppose to the contrary that $Y_1 \neq \{y_1\}$. Then $|Y_1| \geq 2$. Note that $\Delta(G') \leq \Delta(G) = 3$. Since G' is 2-connected and $\delta_{G'}(Y_1) = \{e_{11}, e_{21}\}$, the ends of e_{11} and e_{21} in Y_1 are different. Let C_4 be a circuit in G' containing all the edges in $\{e_{11}, e_{12}, e_{22}, e_{21}\}$ such that $|V(C_4) \cap V_2(G)|$ is as large as possible. Since $G[Y_1]$ is balanced and $|\delta_G(Y_1)| = 3$, with a similar argument in (iv), C_4 contains a 2-vertex in Y_1 and also contains the 2-vertex y_2. Thus C_4 contains at least two 2-vertices and hence is removable. Since $\delta_G(Y_1) \cap E(C_4) = \{e_{11}, e_{21}\}$ and $|\delta_G(Y_1)| = 3$, $G - V(C_4)$ is balanced. Since C_4 does not contain e_0, the only negative edge, C_4 is balanced, meaning $\sigma(C_4) = 1 = \epsilon$, a contradiction to Claim 2-(2b). This completes the proof of (v).

Let x_{11}, x_{12} and x_{13} be the ends of e_{11}, e_{12} and e_{13} in X_1, respectively. By (S4), $G[\{x_{12}, x_{13}, x_2, y_2, y_3\}]$ is not a 4-circuit, so $x_{12} \neq x_{13}$. Together with (iii), (iv), and (v), the structure of G' is shown in Fig. 3.

Now we can complete the proof of Claim 8.

Recall that $G'[X_1]$ is connected. If there is an (x_{12}, x_{13})-path P in $G'[X_1]$ containing y_0, then $C_5 = P \cup \{e_{12}, e_{22}, e_{23}, e_{13}\}$ is a circuit containing y_0 and two 2-vertices y_2, y_3, a contradiction to (i). Hence by Menger's Theorem, $G'[X_1] = G[X_1]$ has a cut-edge separating y_0 from $\{x_{12}, x_{13}\}$.

Let B_1 be the maximal 2-connected subgraphs in $G[X_1]$ containing y_0. Then every edge in $\delta_{G[X_1]}(V(B_1))$ is a cut-edge of $G[X_1]$ by the maximality of B_1. Since $G[X_1]$ has a cut-edge separating y_0 from $\{x_{12}, x_{13}\}$, x_{12} and x_{13} are in the same component, denoted by B_2, of $G[X_1] - V(B_1)$. Since G' is 2-connected and $\delta_{G'}(X_1) = \{e_{11}, e_{12}, e_{13}\}$, $x_{11} \notin V(B_2)$. Let $\delta_{G[X_1]}(V(B_2)) = \{e'\}$ and z be the end of e' in B_2. Then there exists an (x_{11}, z)-path P' in $G'[X_1]$ containing y_0.

Recall that $x_{12} \neq x_{13}$. WLOG assume $z \neq x_{13}$. Since $\delta_G(V(B_2)) = \{e_{12}, e_{13}, e'\}$ and B_2 is balanced and has at least two vertices, by Claim 4, B_2 has a (z, x_{13})-path P'' containing at least one vertex in $V_2(G)$. Then $C_6 = P' \cup P'' \cup x_{13}y_3x_2y_1x_{11}$ is a circuit containing at least two 2-vertices and y_0, a contradiction to (i). This completes the proof of Claim 8. □

By Claim 8, $\epsilon(G) = |E_N(G)| \geq 2$. Denote $\epsilon(G) = k$. By Claims 1 and 6 and Theorem 6.8, we can choose a minimum subset $S \subseteq E(G) \setminus E_N(G)$ such that $H = G/S$ satisfies the following properties:

(i) $\Delta(H) \leq 3$;
(ii) $H - E_N(H) - \cup_{e \in E_N(H)} Int(P_e)$ is a 2-connected planar graph with a facial circuit C, where P_e is the maximal subdivided edge in H containing e;
(iii) $x_1, \dots, x_k, x_{k+1}, \dots, x_{2k}$ are pairwise distinct and lie in that cyclic order on C, where $E_N(H) = E_N(G) = \{e_1, \dots, e_k\}$ and x_i, x_{k+i} are the two ends of P_{e_i} for each $i \in [1, k]$.

For each $v \in V(H)$, let G_v denote the corresponding component of $G - E(H)$. Note that $\Delta(G_v) \leq \Delta(G) = 3$. By the minimality of S, G_v is 2-connected. Otherwise we choose $S \setminus S_v$ to replace S, where S_v is the set of cut-edges of G_v. Moreover, $S = \cup_{v \in V(H)} E(G_v)$ and $E(G) = E(H) \cup S$.

Claim 9. *$k = 2$ and $|Int(P_{e_1})| + |Int(P_{e_2})| = 1$.*

Proof of Claim 9. Since $k \geq 2$, it is easy to see $H - \{x\}$ contains an unbalanced theta for any vertex x with $d_H(x) = 2$. Thus by Claim 3 and by the minimality of S, we have that if $d_H(x) = 2$ then $G_x = \{x\}$.

We construct a circuit C_H in the following cases. If there are distinct $i, j \in [1, k]$ such that $|Int(P_{e_i})| = |Int(P_{e_j})| = 0$, let $C_H = C$; If $|Int(P_{e_i})| + |Int(P_{e_{i+1}})| \geq 2$ for some $i \in [1, k]$, let $C_H = C - E(x_i C x_{i+1}) - E(x_{i+k} C x_{i+k+1}) + P_{e_i} + P_{e_{i+1}}$. Note that G_v is 2-connected for any $v \in V(H)$, $\Delta(H) \leq 3$ and $\Delta(G) = 3$. Then C_H can be extended to a removable circuit C_G of G such that $\sigma(C_G) = 1 = \epsilon$ and $G - V(C_G)$ is also balanced, a contradiction to Claim 2-(2b). This completes the proof of the claim. □

WLOG assume that $Int(P_{e_1}) = \emptyset$ and $Int(P_{e_2}) = \{y\}$ by Claim 9. Then $P_{e_1} = x_1 x_3$ and $P_{e_2} = x_2 y x_4$. Denote $A_i = x_i C x_{i+1} \pmod 4$ for $i \in [1, 4]$, $C_1 = P_{e_1} \cup A_1 \cup P_{e_2} \cup A_3$, and $C_2 = P_{e_1} \cup A_4 \cup P_{e_2} \cup A_2$. Note that both C_1 and C_2 contain the 2-vertex y. See Fig. 4.

Claim 10. *$H = G$ and $V_2(G) = \{y\}$.*

Proof of Claim 10. As noted in the proof of Claim 9, for each x with $d_H(x) = 2$, $G_x = \{x\}$. In particular, $G_y = \{y\}$.

Note that G_x is balanced and $|\delta_G(G_x)| \leq 3$ for every $x \in V(H)$. Thus by Claim 4, we have the following fact:

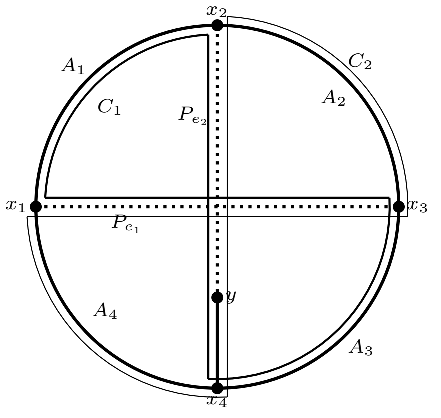

Fig. 4. C_1 and C_2 in $C \cup P_{e_1} \cup P_{e_2}$.

(a) If G_x is nontrivial, then for each two distinct ends u, v in $V(G_x)$ of $\delta_G(G_x)$, there is an (u, v)-path in G_x containing at least one vertex in V_2.

Let $x \in V(C)$. WLOG assume $x \in V(C_1)$. Note that if $d_H(x) = 2$, then $d_G(x) = 2$. Thus, if $d_H(x) = 2$ or if G_x is nontrivial, C_1 can be extended to a circuit C_1' of G such that C_1' contains the 2-vertex y and one 2-vertex in G_x (the latter case follows from (a)). Hence C_1' is removable, $\sigma(C_1') = 1 = \epsilon$, and $G - V(C_1')$ is balanced, a contradiction to Claim 2-(2b). Therefore $d_H(x) = 3$ and $G_x = \{x\}$ for each $x \in V(C)$.

Next we show that y is the only 2-vertex in G. Suppose to the contrary that u is a 2-vertex in G. Then $u \notin V(C)$. Since G is 2-connected, there are two internally disjoint (u, C)-paths Q_1 and Q_2 in G with v_1 and v_2 the end vertices in C respectively. Since $\Delta(G) = 3$, $v_1 \neq v_2$. Let $C_3 = Q_1 \cup Q_2 \cup v_1Cv_2$ and $C_4 \in \{C_1, C_2\}$ such that $V(C_4) \cap \{v_1, v_2\} \neq \emptyset$. Then $C' = C_3 \Delta C_4$ is a circuit containing two 2-vertices $\{y, u\}$ and the two negative edges. Thus C' is removable, $\sigma(C_1') = 1 = \epsilon$, and $G - V(C')$ is balanced, which contradicts Claim 2-(2b). Thus $V_2(G) = \{y\}$.

Since $V_2(G) = \{y\}$, G_x is trivial by (a). Therefore $H = G$. □

Claim 11. *$Int(A_i) \neq \emptyset$ for each $i \in [1, 4]$.*

Proof of Claim 11. Suppose to the contrary that $Int(A_i) \neq \emptyset$ for some $i \in [1, 4]$. WLOG assume $Int(A_1) = \emptyset$. Then A_1 is a chord in $\mathcal{U}(C_2)$. Since C_2 contains the 2-vertex y, C_2 is removable, which contradicts Claim 2-(2b) since $\sigma(C_2) = 1 = \epsilon$ and $G - V(C_2)$ is balanced. □

The final step.

By Claim 11, let $y_1 \in Int(A_1)$ be the neighbor of x_1. Let Q be the component of $G - E(C)$ containing y_1. Since $d_G(y_1) = 3$ by Claim 10, Q is nontrivial. Obviously, $V(Q) \cap \{x_1, x_2, x_3, x_4\} = \emptyset$ since $\Delta(G) = 3$.

If there is a vertex y_2 in $V(Q) \cap (Int(A_2) \cup Int(A_3))$, let P be a (y_1, y_2)-path in Q. Since $\Delta(G) \leq 3$, $C_3 = P \cup y_1 C y_2$ is a circuit containing x_2. Then $C' = C_2 \triangle C_3$ is a circuit of G containing y and the chord $x_1 y_1 \in \mathcal{U}(C')$. Thus C' is a removable circuit of G, a contradiction to Claim 2-(2b) since $G - V(C')$ is balanced.

If $V(Q) \cap (Int(A_2) \cup Int(A_3)) = \emptyset$, then $V(Q) \cap V(C) \subseteq Int(A_4) \cup Int(A_1)$. Note that $|V(Q) \cap V(C)| \geq 2$ since G is 2-connected. Let $y_2, y_3 \in V(Q) \cap V(C)$ be two ends of a segment P' of $A_4 \cup A_1$ such that the length of P' is as large as possible. By Claim 10, $G' = G - x_1x_3 - y$ is a 2-connected planar graph with a facial circuit C, and so $T' = \delta_{G'}(V(P')) \cap E(C)$ is a 2-edge-cut of G'. Let $T = T'$ if $y_2, y_3 \in Int(A_1)$, and otherwise $T = T' \cup \{x_1x_3\}$. Then T is an edge-cut of G with $|T| \leq 3$ and the component, denoted by R, of $G - T$ containing y_2 is balanced and doesn't contain y. Since $|\delta_G(V(R))| = |T| \leq 3$, by (S3), $\sum_{v \in V(R)}(3 - d_G(v)) \geq 4 - |\delta_G(V(R))| - 2|\mathcal{U}(R)| \geq 1$, and so this component R contains a 2-vertex (distinct from y), which contradicts $V_2(G) = \{y\}$ by Claim 10. This completes the proof of Lemma 6.13. □

6.3. Completing the proof of Theorem 1.3

Finally we are to complete the proof of Theorem 1.3 in this subsection.

By Lemma 6.6, it suffices to show that every cubic shrubbery G admits a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF. If G is balanced, then such a flow exists by Theorem 6.2.

Assume that G is unbalanced. We claim that G contains either an unbalanced theta or a negative loop.

If G is 2-connected, then for any unbalanced circuit C, we can easily find a path in $G - E(C)$ to connect two distinct vertices of $V(C)$, and thus G has an unbalanced theta.

If G is not 2-connected, then it has an cut-edge since G is cubic. Let B be a leaf block of G. If B is trivial, then B is a negative loop. If B is nontrivial, then B is unbalanced by Proposition 2.2 since G is flow-admissible by (S2). Since B is 2-connected and all vertex except one has degree 3, similar to the argument in the case when G is 2-connected, one can find an unbalanced theta in B, which is also an unbalanced theta in G.

By the claim, we apply Lemma 6.13 on cubic shrubbery G with $\varepsilon = 1$ to obtain an NZF $f = (f_1, f_2)$ with $\sigma(\text{supp}(f_1)) = \varepsilon = 1$. By Definition 6.10 this is a balanced $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF as desired. This completes the proof of Theorem 1.3.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM, vol. 244, Springer, 2008.
[2] A. Bouchet, Nowhere-zero integral flows on a bidirected graph, J. Comb. Theory, Ser. B 34 (1983) 279–292.
[3] J. Cheng, Y. Lu, R. Luo, C.-Q. Zhang, Signed graphs: from modulo flows to integer-valued flows, SIAM J. Discrete Math. 32 (2018) 956–965.
[4] R. Diestel, Graph Theory, fourth edn., Springer-Verlag, 2010.
[5] H. Fleischner, Eine gemeinsame Basis für die Theorie der eulerschen Graphen und den Satz von Petersen, Monatshefte Math. 81 (1976) 267–278.
[6] F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs — a nonhomogeneous analogue of nowhere-zero flow properties, J. Comb. Theory, Ser. B 56 (1992) 165–182.

[7] Y. Lu, R. Luo, C.-Q. Zhang, Multiple weak 2-linkage and its applications on integer flows on signed graphs, Eur. J. Comb. 69 (2018) 36–48.
[8] P.D. Seymour, Nowhere-zero 6-flows, J. Comb. Theory, Ser. B 30 (1981) 130–135.
[9] W.T. Tutte, A contribution to the theory of chromatic polynomials, Can. J. Math. 6 (1954) 80–91.
[10] M.E. Watkins, D.M. Mesner, Cycles and connectivity in graphs, Can. J. Math. 19 (1967) 1319–1328.
[11] D.B. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, NJ, 1996.
[12] R. Xu, C.-Q. Zhang, On flows in bidirected graphs, Discrete Math. 299 (2005) 335–343.
[13] O. Zýka, Nowhere-zero 30-flow on bidirected graphs, Thesis, KAM-DIMATIA, Series 87-26, Charles University, Praha, 1987.