Hamiltonian Weights and Unique 3-Edge-Colorings of Cubic Graphs

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ABSTRACT

A (1,2)-eulerian weight w of a graph is hamiltonian if every faithful cover of w is a set of two Hamilton circuits. Let G be a 3-connected cubic graph containing no subdivision of the Petersen graph. We prove that if G admits a hamiltonian weight then G is uniquely 3-edge-colorable.

1. INTRODUCTION

Most standard graph-theoretic terms that are used in this paper can be found for instance in [3]. All graphs we considered in this paper may have multiple edges but no loops. A *(1, 2)-eulerian weight* w of a 2-connected graph G is a weight w: $E(G) \mapsto \{1, 2\}$ such that the total weight of each edge-cut is even. A *faithful cover* of w is a family C of circuits such that each edge e is contained in precisely $w(e)$ circuits of C. The topic of faithful coverings of eulerian weights has attracted many mathematicians (see survey papers [6–9,19], etc.). Some well-known conjectures on this topic (such as the circuit double cover conjecture due to Szekeres [12] and, Seymour [11]) still remain open. A cubic graph is *uniquely 3-edge-colorable* if G has precisely one 1-factorization. The topic of uniquely 3-edge-colorable cubic graphs is also a very interesting topic in graph theory (see [13, 14, 17, 4, and 10], etc.). Some well-known conjectures (such as every planar, uniquely 3-edge-colorable, cubic graph has a triangle due to Fiorini and Wilson [4]) still remain open. In this paper, we will study the relations between the faithful coverings and the uniquely 3-edge-colorability of cubic graphs.

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Let w be a (1,2)-eulerian weight of a cubic graph G. A faithful cover C of w is *hamiltonian* if C is a set of two Hamilton circuits. A (1,2)-eulerian weight w of G is *hamiltonian* if *every* faithful cover of w is hamiltonian. Let G be a uniquely 3-edge-colorable cubic graph with the 1-factorization $\mathcal{F} = \{F_1, F_2, F_3\}$. Let w: $E(G) \to \{1,2\}$ such that

$$w(e) = \begin{cases} 2 & \text{if } e \in F_3, \\ 1 & \text{if } e \in F_1 \cup F_2. \end{cases}$$

It is obvious that w has a hamiltonian cover $\{F_1 \cup F_3, F_2 \cup F_3\}$. It is natural to ask the following question: Let G and w be defined as above. Is every faithful cover of w hamiltonian (i.e., is w a hamiltonian weight)? The answer is no. A uniquely 3-edge-colorable graph $P(9,2)$ (see Figure 1) constructed by Tutte ([17]) does not admit a hamiltonian weight. (A graph $P(n,k)$, called a *generalized Pertersen graph,* is defined as follows: $P(n,k)$ has $2n$ vertices, namely $v_0, .v_{n-1}., u_0, ..., u_{n-1}$, the vertex v_i is joined to v_{i+1}, v_{i-1} and u_i,u_i is further joined to u_{i+k} and u_{i-k} (where addition is mod$n$).)

The 1-factorization $\mathcal{F} = \{F_0, F_1, F_2\}$ of $P(9,2)$ is

$$F_1 = \{v_2u_2, v_5u_5, v_8u_8, v_3v_4, v_6v_7, v_0v_1, u_3u_7, u_4u_0, u_6u_1\},$$
$$F_2 = \{v_0u_0, v_3u_3, v_6u_6, v_1v_2, v_4v_5, v_7v_8, u_1u_5, u_2u_7, u_4u_8\},$$

and

$$F_3 = \{v_1u_1, v_4u_4, v_7u_7, v_2v_3, v_5v_6, v_8v_0, u_2u_6, u_3u_8, u_5u_0\}.$$

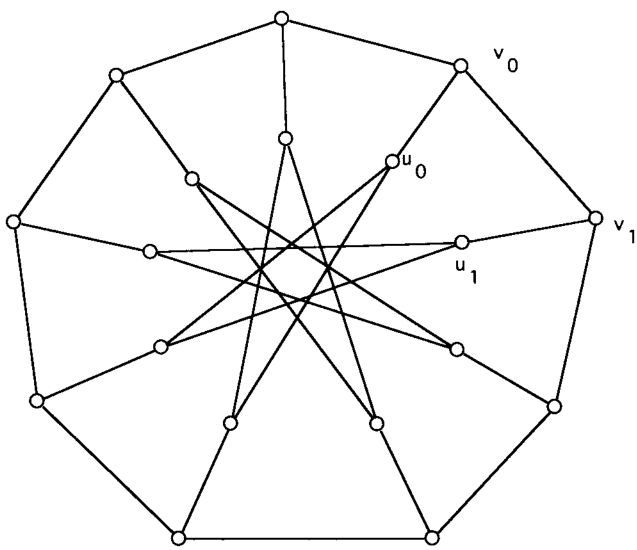

FIGURE 1. Tutte's graph $P(9,2)$.

For the eulerian weight

$$w(e) = \begin{cases} 2 & \text{if } e \in F_3, \\ 1 & \text{if } e \in F_1 \cup F_2, \end{cases}$$

the graph P(9.2) has a non-Hamiltonian faithful cover $C = \{C_1, C_2, C_3, C_4\}$, where

$$C_1 = v_2v_3v_4u_4u_0u_5v_5v_6u_6u_2v_2,$$

$$C_2 = v_0u_0u_5u_1v_1v_2v_3u_3u_8v_8v_0,$$

$$C_3 = v_0v_1u_1u_6u_2u_7v_7v_8v_0,$$

and

$$C_4 = v_7u_7u_3u_8u_4v_5v_6v_7.$$

The study of hamiltonian weights and uniquely 3-edge-colorable cubic graphs are motivated by the circuit double cover conjecture that *every 2-edge-connected graph has a family of circuits that covers each edge precisely twice.* Let a cubic graph G be a minimal counterexample to the circuit double cover conjecture. For each $e_0 \in E(G)$, the graph $G \backslash e_0$ has a circuit double cover. Choose a circuit double cover C such that C contains the maximum number of circuits. Let C_1, $C_2 \in C$ such that $E(C_1) \cap E(C_2) \neq \phi$. Let H be the underlying (cubic) graph of the induced subgraph $G[E(C_1) \cup E(C_2)]$. Let w be the $(1, 2)$-eulerian weight on $E(H)$ such that $w(e)$ is the number of circuits of $\{C_1, C_2\}$ containing the edge e, for each edge $e \in E(H)$. It is not hard to see that $\{C_1, C_2\}$ is a hamiltonian cover of w and w is a hamiltonian weight of H. Hopefully, some results about cubic graphs admitting hamiltonian weight will provide some new tools to attack the circuit double cover conjecture.

2. LEMMAS AND THEOREMS

A subgraph H of a graph G is *even* if the degree of each vertex of H is even. Let w be a $(1, 2)$-eulerian weight of G. Denote $E_i = \{e \in E(G): w(e) = i\}$. Since w is an eulerian weight, it is obvious that the subgraph of G induced by E_1 is an even subgraph of G. The following lemma is straightforward.

Lemma 2.1. Let G be a cubic graph with a 1-factorization $\mathcal{F} = \{F_1, F_2, F_3\}$ and w be a $(1, 2)$-eulerian weight of G. Then

(i) The set of even subgraphs $\{L_{ij} : \{i, j\} \subset \{1, 2, 3\}$ and $i \neq j\}$ covers each edge $e \in E(G)$ precisely $w(e)$ times, where $L_{ij} = E_1 \Delta (F_i \cup F_j)$.

(ii) One of $\{L_{12}, L_{13}, L_{23}\}$ is empty if and only if $E_2 \in \mathcal{F}$.

Theorem 2.2. Let G be a cubic graph admitting a hamiltonian weight w. Then the following statements are equivalent:

(i) G is uniquely 3-edge-colorable.
(ii) G has precisely three Hamilton circuits.
(iii) the hamiltonian weight w has precisely one faithful cover.
(iv) E_1 is a Hamilton circuit of G.

Proof: (i) $\Rightarrow$ (ii). It is well known and easy to prove (even without the assumption of admitting a hamiltonian weight).

(ii) $\Rightarrow$ (iii). If the hamiltonian weight w has at least two distinct hamiltonian covers, then G has at least four distinct Hamilton circuits, which is a contradiction.

(iii) $\Rightarrow$ (iv). Let $\{H_1, H_2\}$ be the unique hamiltonian cover of w. Let C be a component of E_1. If C is not a Hamilton circuit of G, then $\{H_1 \Delta C, H_2 \Delta C\}$ is also a faithful cover of w and distinct from $\{H_1, H_2\}$. This contradicts that w is a hamiltonian weight with only one faithful cover.

(iv) $\Rightarrow$ (i). Since E_1 is a Hamilton circuit and E_2 is a 1-factor, G has a 1-factorization $\mathcal{F}_1 = \{F_1, F_2, F_3\}$, where $F_1 \cup F_2 = E_1$ and $F_3 = E_2$. Assume that G is not uniquely 3-edge-colorable. Let $\mathcal{F}_2 = \{F_4, F_5, F_6\}$ be a 1-factorization distinct from $\mathcal{F}_1$. By (i) of Lemma 2.1, the set of even subgraphs

$$\mathcal{C} = \{E_1 \Delta (F_1 \cup F_j): \{i, j\} \subset \{4, 5, 6\} \text{ and } i \neq j\}$$

covers each edge $e \in E(G)$ precisely $w(e)$ times. Thus, the set of circuits of circuit decompositions of the members of $\mathcal{C}$ is a faithful circuit cover of w. Since w is a hamiltonian weight, one member of $\mathcal{C}$ is empty. By (ii) of Lemma 2.1, $E_2 \in \mathcal{F}_2$. Without loss of generality, let $E_2 = F_6$. Therefore, $F_6 = F_3$ and $F_4 \cup F_5 = E_1 = F_1 \cup F_2$. Since E_1 is a Hamilton circuit, we have that $\{F_1, F_2\} = \{F_4, F_5\}$ and this is a contradiction since $\mathcal{F}_2$ is a 1-factorization distinct from $\mathcal{F}_1$. Then G is uniquely 3-edge-colorable. ∎

Note that (i) and (ii) are not always equivalent (see [13]) without the assumption that G admits a hamiltonian weight. It was conjectured by Greenwell and Kronk ([5], also see [13]) that if a cubic graph G has exactly three Hamilton circuits, then G is uniquely 3-edge-colorable. This conjecture was disproved by A. Thomason ([13]), who found a family of counterexamples: $P(6k + 3, 2)$ for $k \geq 2$, each of which has exactly three Hamilton circuits, but is not uniquely 3-edge-colorable. By Theorem 2.2, those graphs constructed by Thomason do not admit hamiltonian weights.

Lemma 2.3 (Alspach and Zhang [1] or see [2]). Let G be a 2-connected cubic graph containing no subdivision of the Petersen graph. Then G has a faithful cover for every $(1, 2)$-eulerian weight of G.

Theorem 2.4. If a 3-connected cubic graph G admits a hamiltonian weight and contains no subdivision of the Petersen graph, then G is uniquely 3-edge-colorable.

Proof. Let w be a hamiltonian weight of G and $\{H_1, H_2\}$ be a hamiltonian cover of w.

1. Each component of the 2-factor $E_1 = H_1 \Delta H_2$ is a circuit of even length since $\{H_1, H_2\}$ induces a 3-edge-coloring $\{H_1 \backslash H_2, H_2 \backslash H_1, H_1 \cap H_2\}$ of G.
2. For each weight two edge $e_0 = xy$, by Lemma 2.3, $G' = G \backslash \{e_0\}$ has a faithful cover C with respect to the restriction of w to $E(G')$. Define an auxiliary graph $A(C)$ with the vertex set C and two vertices C_i and C_j re adjacent in $A(C)$ if and only if the corresponding circuits C_i and C_j have a nonempty intersection. A circuit chain $\mathcal{P} = C_1 \cdots C_r$ joining the vertices x and y of G is a shortest path in $A(C)$ joining a circuit C_1 containing x and a circuit C_r containing y.

Let H be the graph induced by edges covered by circuits of $\mathcal{P}$ and the edge e_0, and let w' be a $(1,2)$-eulerian weight on $E(H)$ such that $w'(e)$ is the number of circuits of $\mathcal{P}$ containing the edge e, and $w'(e_0) = 2$. By Lemma 2.3, H has a faithful cover C'. Then $C' \cup [C \backslash \mathcal{P}]$ is a faithful cover of G. If $C \neq \mathcal{P}$, then the faithful cover $C' \cup [C \backslash \mathcal{P}]$ of w is not hamiltonian. This contradicts that w is hamiltonian and therefore $C = \mathcal{P}$.

Color the edges of $[E(C_1) \cup E(C_3) \cdots] \backslash [E(C_2) \cup E(C_4) \cdots]$ red, the edges of $[E(C_2) \cup E(C_4) \cdots] \backslash [E(C_1) \cup E(C_3) \cdots]$ blue and the edges contained in the intersections of two circuits of C yellow. The graph H is 3-edge-colored. Here each component of E_1 is colored red and blue. Since x and y are the only degree two vertices of H, each component of E_1 containing neither x nor y is a circuit of even length. If the vertices x and y are contained in distinct circuits of E_1, then E_1 has two components of odd lengths. But by 1, every component of E_1 is of even length. Therefore, x and y must be in the same component of E_1.

Since $e_0 = xy$ is an arbitrary weight two edge of G, we have that every weight two edge of G must join two vertices on the same component of E_1. Thus E_1 has only one component, which is, therefore, a Hamilton circuit of G. By (iv) of Theorem 2.2, G is uniquely 3-edge-colorable. ∎

Theorem 2.5. Let G be cubic graph. If G admits at least two hamiltonian weights, then G is uniquely 3-edge-colorable.

Proof. Let w and w' be two distinct hamiltonian weights of G and $\{H_1, H_2\}$, $\{H_1', H_2'\}$ be hamiltonian covers of w and w', respectively. Both hamiltonian covers induce 1-factorizations $\mathcal{F} = \{H_1 \backslash H_2, H_2 \backslash H_1, H_1 \cap H_2\}$ and $\mathcal{F}' = \{H_1' \backslash H_2', H_2' \backslash H_1', H_1' \cap H_2'\}$. Denote $E_{w=i}$ (and $E_{w'=i}$), the set of edges e of G with weight $w(e) = i$ (and $w'(e) = i$, respectively), for $i = 1, 2$. Since w and w' are two distinct eulerian weights, $E_{w=2} \neq E_{w'=2}$. By

(ii) of Lemma 2.1, $H_1 \cap H_2 = E_{w=2} \in \mathcal{F}'$ and $H_1' \cap H_2' = E_{w'=2} \in \mathcal{F}$. Without loss of generality, we assume that $H_1 \cap H_2 = E_{w=2} = H_1' \backslash H_2'$. Thus $E_{w=1} = H_2'$ is a Hamilton circuit of G. By (iv) of Theorem 2.2, G is uniquely 3-edge-colorable. ∎

3. Δ− Y OPERATION AND EQUIVALENCE CLASSES OF CUBIC GRAPHS

Definition. The $\Delta - Y$ operation of a cubic graph is either (i) contracting the edges of a triangle, or (ii) replacing a vertex of the graph by a triangle (see Figure 2).

Let $\mathcal{T}$ be a graph with the vertex set $V(\mathcal{T})$ being the collection of all connected cubic graphs. Two vertices G_1 and $G_2 \in V(\mathcal{T})$ are adjacent in $\mathcal{T}$ if and only if G_1 can be obtained from G_2 by a $\Delta - Y$ operation. $V(\mathcal{T})$ has a partition into equivalence classes: each class is a component of the graph $\mathcal{T}$. That is, $G_1 \equiv G_2$ if and only if G_1 can be obtained from G_2 by a series of $\Delta - Y$ operations. In each class, there is only one triangle-free element that is the element with the least number of vertices in the class. Denote by T_1 the class containing K_2^3, which is the graph with two vertices and three parallel edges, and by T_2 the class containing $P(9,2)$.

Remarks. The $\Delta - Y$ operation preserves each of the following properties:

1. planarity,
2. 3-edge-colorability,
3. number of 1-factorizations,
4. number of Hamilton circuits,
5. number of hamiltonian weights,
6. number of hamiltonian covers of a (1,2)-eulerian weight.

4. CONJECTURES AND REMARKS

In this section, all graphs we consider are 3-connected cubic graphs. Let S_1 be the collection of all uniquely 3-edge-colorable graphs, S_2 be the

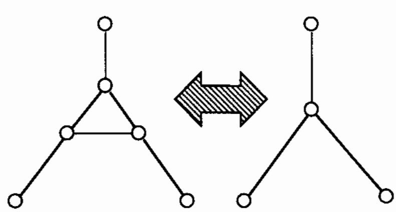

FIGURE 2. $\Delta - Y$ operation.

collections of all graphs admitting hamiltonian weights, PL be the collection of all planar graphs, and $\overline{P}_{10}$ be the collection of all graphs containing no subdivision of the Petersen graph. (Here, $PL \subset \overline{P}_{10}$). It was conjectured by Greenwell and Kronk ([5], also see [13]) that $S_1 = T_1$. (That is, every uniquely 3-edge-colorable cubic graph is planar and has a triangle.) This conjecture was disproved by Tutte, who found the counterexample $P(9,2)$ (see Figure 1). Since the generalized Petersen graph $P(9,2)$ is not planar, the conjecture was later modified as follows,

Conjecture 4.1 (Fiorini and Wilson [4]). Let G be a 3-connected planar cubic graph with at least 4 vertices. If G is uniquely 3-edge-colorable cubic graph, then G has a triangle. That is,

$$PL \cap S_1 = T_1.$$

The author believes that in the construction of a triangle-free, uniquely 3-edge-colorable cubic graph other than K_2^3, a non-3-edge-colorable cubic graph (snark) must be somehow involved. Based on the famous 4-flow conjecture of Tutte ([16]) that no snark belongs to $\overline{P}_{10}$, the author proposes the following conjecture.

Conjecture 4.2. Let G be a 3-connected cubic graph containing no subdivision of the Petersen graph. If G is uniquely 3-edge-colorable, then G must contain a triangle. That is,

$$\overline{P}_{10} \cap S_1 = T_1.$$

Recall Theorem 2.4: $\overline{P}_{10} \cap S_2 \subset S_1$. Note that T_2, which contains $P(9,2)$ is a subset of neither $\overline{P}_{10}$ nor S_2. We propose the following conjecture:

Conjecture 4.3. Let G be a 3-connected cubic graph containing no subdivision of the Petersen graph. If G is uniquely 3-edge-colorable, then G must admit a hamiltonian weight. That is,

$$\overline{P}_{10} \cap S_1 \subset S_2.$$

By Theorem 2.4, we have the following equivalent version of Conjecture 4.3.

Conjecture 4.4. If G is a 3-connected cubic graph containing no subdivision of the Petersen graph, then G admits a hamiltonian weight if and only if G is uniquely 3-edge-colorable. That is,

$$\overline{P}_{10} \cap S_1 = \overline{P}_{10} \cap S_2.$$

Since $T_1 \subset S_2$, Conjecture 4.3, as well as Conjecture 4.4, is implied by Conjecture 4.2. Similar to Conjecture 4.1, we propose

Conjecture 4.5. Every 3-connected cubic graph admitting a hamiltonian weight contains a triangle. That is,

$$S_2 = T_1.$$

The following conjecture is a generalization of Theorem 2.4,

Conjecture 4.6. Every 3-connected cubic graph admitting a hamiltonian weight is uniquely 3-edge-colorable. That is,

$$S_2 \subset S_1.$$

Note that the 3-connectivity in most conjectures of this section cannot be relaxed, since the 2-connected cubic graph H with four vertices $\{a, b, c, d\}$ and the six edges $\{ab, ab, ac, bd, cd, cd\}$ admits a hamiltonian weight w with $E_{w=2} = \{ac, bd\}$ but contains no triangle, and all cubic graphs obtained from H by $\Delta - Y$ operations admit hamiltonian weights but are not uniquely 3-edge-colorable.

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