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Six-flows on almost balanced signed graphs

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Abstract

In 1983, Bouchet conjectured that every flow-admissible signed graph admits a nowhere-zero 6-flow. By Seymour's 6-flow theorem, Bouchet's conjecture holds for signed graphs with all edges positive. Recently, Rollová et al proved that every flow-admissible signed cubic graph with two negative edges admits a nowhere-zero 7-flow, and admits a nowhere-zero 6-flow if its underlying graph either contains a bridge, or is 3-edge-colorable, or is critical. In this paper, we improve and extend these results, and confirm Bouchet's conjecture for signed graphs with frustration number at most two, where the frustration number of a signed graph is the smallest number of vertices whose deletion leaves a balanced signed graph.

KEYWORDS

frustration number, integer flow, signed graph

1 | INTRODUCTION

Tutte [18,19] initiated the study of integer flows as a refinement and a generalization of the face coloring problem of planar graphs. He made three famous conjectures, known as the 5-flow, 4-flow, and 3-flow conjectures. The strongest partial result toward the 5-flow conjecture is the famous 6-flow theorem due to Seymour [16].

Theorem 1.1 (Seymour [16]). *Every bridgeless graph admits a nowhere-zero 6-flow.*

The concept of integer flows on signed graphs naturally comes from the study of graphs embedded on nonorientable surfaces, where nowhere-zero flow emerges as the dual notion to local tension. In 1983, Bouchet [2] proposed the following conjecture.

Conjecture 1.2 (Bouchet [2]). *Every flow-admissible signed graph admits a nowhere-zero 6-flow.*

Bouchet [2] himself proved that every flow-admissible signed graph admits a nowhere-zero 216-flow. Zýka [24] improved the result to 30-flow, and DeVos [3] further improved Zýka's result to 12-flow. Integer flows on signed graphs also have been studied for some specific families of graphs, such as complete and complete bipartite graphs [9], eulerian graphs [10,11], series-parallel graphs [7], Kotzig graphs [14], highly connected graphs [13,20,21], and so forth. Recently, Rollová et al [12] partially confirmed Bouchet's conjecture for signed cubic graphs with at most two negative edges.

Theorem 1.3 (Rollová, Schubert, and Steffen [12]). *Let (G, σ) be a flow-admissible signed cubic graph with two negative edges. Then*

(1) *(G, σ) admits a nowhere-zero 7-flow such that each negative edge has flow value 1.*
(2) *(G, σ) admits a nowhere-zero 6-flow such that each negative edge has flow value 1 if either G contains a bridge, or G is 3-edge-colorable, or G is critical.*

Here, a cubic graph is *critical* if it is not 3-edge-colorable but the resulting graph by deleting any edge admits a nowhere-zero 4-flow.

In this paper, we improve the results in Theorem 1.3.

Theorem 1.4. *Every flow-admissible signed graph with two negative edges admits a nowhere-zero 6-flow such that each negative edge has flow value 1.*

By applying Theorem 1.4, we further confirm Bouchet's conjecture for signed graphs with frustration number at most two. The *frustration number* (resp., *frustration index*) of a signed graph is the smallest number of vertices (resp., edges) whose deletion leaves a balanced graph. Note that, by Lemma 7.6 in [22], *the frustration index is greater than or equal to the frustration number in every signed graph.* Thus, if a signed graph contains exactly two negative edges, then its frustration index (and thus frustration number) is at most two.

Theorem 1.5. *Every flow-admissible signed graph with frustration number at most two admits a nowhere-zero 6-flow.*

Please note that all flows considered in this paper are integer-valued k-flows, not group $\mathbb{Z}_k$-flows.

2 | NOTATION AND TERMINOLOGY

For notation and terminology not defined here we follow [1,23]. Graphs considered in this paper may have multiple edges or loops. *A signed graph* (G, σ) is a graph G associated with a mapping $\sigma : E(G) \to \{\pm 1\}$. G is called the *underlying graph* of (G, σ), and σ is called the *signature* of (G, σ).

An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* if $\sigma(e) = -1$. For a subgraph H of G, we use (H, σ) to represent the signed subgraph $(H, \sigma|_{E(H)})$, where $\sigma|_{E(H)}$ is the restriction of σ on $E(H)$.

A *circuit* is a connected 2-regular graph. A circuit is *balanced* in a signed graph if it contains an even number of negative edges, and *unbalanced* otherwise. A signed graph itself is *balanced* if it does not contain any unbalanced circuit, and is *unbalanced* if it does.

Following Bouchet [2], we view an edge uv of a graph G as two *half edges* h_u and h_v, where h_u is incident with u and h_v is incident with v. Let $H(G)$ be *the set of all half edges of* G and for $u \in V(G)$, let $H_G(u)$ be the set of the half edges incident with u. An *orientation* of (G, σ) is a mapping τ: $H(G) \to \{1, -1\}$ such that for every $uv \in E(G)$, $\tau(h_u)\tau(h_v) = -\sigma(uv)$. For $h_u \in H(G)$, if $\tau(h_u) = 1$, then h_u is oriented away from u; if $\tau(h_u) = -1$, then h_u is oriented toward u.

Definition 2.1. Let (G, σ) be a signed graph associated with an orientation τ. Let k be a positive integer and $f : E(G) \to \mathbb{Z}$ be a mapping.

(1) The *boundary* of f at a vertex v is defined as $\partial f(v) = \sum_{h \in H_G(u)} \tau(h) f(e_h)$, where e_h is the edge of G containing h.
(2) The ordered pair (τ, f) is called an *integer k-flow* (or, simply *k-flow*) of (G, σ) if $\partial f(v) = 0$ for each $v \in V(G)$ and $|f(e)| < k$ for each $e \in E(G)$.
(3) The *support* of f is the set of all edges of G with $f(e) \neq 0$ and is denoted by $\text{supp}(f)$. A flow (τ, f) is *nowhere-zero* if $\text{supp}(f) = E(G)$.

For the sake of convenience, a nowhere-zero integer flow (resp., nowhere-zero k-flow) is abbreviated as *an NZF* (resp., *a k-NZF*). Observe that a signed graph admits a k-NZF under some orientation τ if and only if it admits a k-NZF under any orientation τ'.

A signed graph is *flow-admissible* if it admits a k-NZF for some positive integer k. Refining the results in [2] or [13], we have the following characterization: *a signed graph (G, σ) is flow-admissible if and only if for any $e \in E(G)$, the number of balanced components of $(G - e, \sigma)$ is less than or equal to that of (G, σ).*

Assume that (G, σ) is a signed graph with an orientation τ and $e = uv \in E(G)$. By the definition of τ, if $e = uv$ is positive, then h_u and h_v are directed both from u to v, or both from v to u. Thus, if all edges of (G, σ) are positive, then every integer flow on (G, σ) is also an integer flow of G. In this sense, integer flows on signed graphs generalize the concept of integer flows on ordinary graphs.

In a signed graph, *switching* at a vertex u means reversing the signs of all edges incident with u. In particular, a signed graph is balanced if and only if all of its edges can be changed to positive via a sequence of switching operations. We also note that the existence of a k-NZF, frustration number, and frustration index (see [22]) of a signed graph are invariants under the switching operation.

3 | PROOF OF THEOREM 1.4

3.1 | Preliminaries

Let H be a graph and C be a circuit. In [16], Seymour defined an operation as follows:

$$\Phi_k\text{: add the circuit } C \text{ into } H \text{ if } |E(C) \backslash E(H)| \leq k.$$

For a subgraph H of G, denote by $\langle H\rangle_k$, the maximum subgraph of G obtained from H via Φ_k-operations. In the same paper, Seymour proved the following results and thus obtained the famous 6-flow theorem.

Lemma 3.1 (Seymour [16]). *Let G be a graph with an orientation τ, and H be a subgraph of G. If $\langle H\rangle_2 = G$, then G admits a 3-flow (τ, f) such that $E(G)\backslash E(H) \subseteq \text{supp}(f)$.*

To *contract* an edge e of a graph G is to delete the edge and then identify its ends. The resulting graph is denoted by G/e. For $S \subseteq E(G)$, let G/S denote the graph obtained from G by contracting all edges of S. For any $U \subseteq V(G)$, let $\overline{U} = V(G)\backslash U$, and use $[U, \overline{U}]_G$ or $[U, \overline{U}]$ to denote the set of edges between U and $\overline{U}$. If $U = \{u\}$, we simply abbreviate $[\{u\}, \overline{\{u\}}]$ as $E_G(u)$.

The following lemma is implied by the proof of Lemma 3.2 in [16]. For more details, we refer the reader to Lemma 5.3.5 in the book [23]. Here, the majority of the arguments in our proof is from their proofs.

Lemma 3.2. *Let G be a graph and H be a connected subgraph of G such that $G/E(H)$ is 3-edge-connected. Then, there is a set of edge-disjoint circuits $C_1,\ldots,C_t$ of $G - E(H)$ such that $\langle H \cup C_1 \cup\cdots\cup C_t\rangle_2 = G$.*

Proof. Since H is connected, we can choose a set of edge-disjoint circuits $C_1,\ldots,C_r$ of $G - E(H)$ ($r = 0$ possibly) such that

(i) $\langle H \cup C_1 \cup \cdots\cup C_r\rangle_2$ is connected;
(ii) subject to (i), r is as large as possible.

Let $X = \langle H \cup C_1 \cup \cdots\cup C_r\rangle_2$. Assume that $G - X$ is not empty and let Q be a component of $G - X$. If Q has a bridge, then choose a bridge e such that one component Q' of $Q - \{e\}$ is as small as possible. If Q has no bridge, then simply let $Q' = Q$. Since $G/E(H)$ is 3-edge-connected, there are two distinct edges $uu', vv' \in [V(Q'), V(X)]$, where $u, v \in V(Q')$, $u \neq v$, and $u', v' \in V(X)$. Since Q' has no bridge, Q' has two edge-disjoint paths P_1, P_2 jointing u and v. Then $X' = \langle H \cup C_1 \cup\cdots\cup C_r \cup (P_1 \cup P_2)\rangle_2$ and X' is connected where $C_1,\ldots,C_r$, $P_1 \cup P_2$ are edge-disjoint circuits of $G - E(H)$. This contradicts the maximality of r . □

An unpublished manuscript [3] of DeVos contains an extension lemma on modular flows. By applying this lemma, Lu, Luo, and Zhang extended it to integer flows in the following lemma.

Lemma 3.3 (Lu, Luo, and Zhang [8]). *Let k be a positive integer, and let G be a graph with an orientation τ and admitting a k-NZF. If a vertex v of G is of degree at most three and g: $E_G(u) \to \{\pm 1,\ldots,\pm(k-1)\}$ satisfies $\partial g(u) = 0$, then there is a k-NZF (τ, f) on G such that $f|_{E_G(u)} = g$.*

Lemma 3.4 (Thomassen [17] and Seymour [15]). *Let e_1, e_2 be two distinct edges of a connected graph G. Then, the following statements are equivalent.*

(1) *G does not contain a pair of edge-disjoint circuits C_1 and C_2 of G such that $e_i \in E(C_i)$ for $i = 1, 2$.*

(2) *There is an edge subset $S \subseteq E(G)\backslash\{e_1, e_2\}$ such that G/S is a connected subcubic graph, which can be drawn in the plane with exactly one crossing pair $\{e_1, e_2\}$.*

Lemma 3.5. *Let (G, σ) be a 2-connected unbalanced signed graph with frustration index 2 and $\sigma^{-1}(-1) = \{e_1, e_2\}$. Then G does not contain a pair of edge-disjoint unbalanced circuits if and only if there is a subset $S \subseteq E(G)\backslash\{e_1, e_2\}$ such that $(G/S, \sigma)$ is a connected cubic signed graph, which can be drawn in the plane with exactly one crossing pair $\{e_1, e_2\}$.*

Proof. Since (G, σ) contains exactly two negative edges e_1 and e_2, every unbalanced circuit in (G, σ) contains exactly one negative edge. So (G, σ) contains a pair of edge-disjoint unbalanced circuits if and only if G has a pair of edge-disjoint circuits C_1 and C_2 such that $e_i \in E(C_i)$ for $i = 1, 2$. Thus, the lemma follows from Lemma 3.4. □

We also need the following results.

Theorem 3.6 (Harary [5]). *A signed graph is balanced if and only if its vertex set can be partitioned into two sets (either of which may be empty) in such a way that each edge between the sets is negative and each edge within a set is positive.*

Lemma 3.7 (Jaeger [6], or see Exercise 3.23 in [23], Lemma 2.1 in [4]). *Let G be a bridgeless cubic graph, drawn in the plane with at most one crossing. Then G is 3-edge-colorable.*

3.2 | Proof of Theorem 1.4

We prove the theorem by contradiction. Let (G, σ) be a counterexample with minimum $|E(G)|$ and $\sigma^{-1}(-1) = \{e_1, e_2\}$. Since (G, σ) is flow-admissible, e_1 and e_2 must be contained in the same component of G, and thus G is connected by the minimality of $|E(G)|$. By the minimality of $|E(G)|$ again, positive edges do not contain a digon.

Claim 1. (G, σ) is unbalanced.

Proof of Claim 1. Suppose to the contrary that (G, σ) is balanced. By Theorem 1.1, it admits a 6-NZF. Hence, we only need to show that $f(e_1) = f(e_2) = 1$ for some 6-NZF (τ, f). By Theorem 3.6, since (G, σ) is balanced and $\sigma^{-1}(-1) = \{e_1, e_2\}$, G contains a vertex subset U such that $[U, \overline{U}] = \{e_1, e_2\}$. Since the balanced graph (G, σ) is flow-admissible, G is 2-edge-connected and both induced subgraphs $G[U]$ and $G[\overline{U}]$ are connected.

Let G' be the graph obtained from G by replacing $e_1 = u_1v_1$ with a path u_1uv_1. Fix an orientation τ_0 on u_1uv_1 such that it is a directed path, and define g_0: $\{u_1u, uv_1\} \mapsto \{1\}$. By Lemma 3.3 and Theorem 1.1, (τ_0, g_0) can be extended to a 6-NZF (τ_1, g_1) on G' with $g_1(u_1u) = g_1(uv_1) = 1$. Note that $\{u_1u, e_2\}$ is also a 2-edge-cut of G' since $\{e_1, e_2\}$ is a 2-edge-cut of G. So $|g_1(e_2)| = g_1(u_1u) = 1$. Clearly, (τ_1, g_1) can be adjusted to a 6-NZF (τ_2, f) such that $f(e_1) = f(e_2) = 1$.

Let τ be the orientation of (G, σ) obtained from τ_2 by reversing the direction of every half edge whose end is in U. Then (τ, f) is a desired 6-NZF, which contradicts that (G, σ) is a counterexample. □

Claim 2. For any $U \subseteq V(G)$ with $|E(G[U])| \geq 1$ and $|E(G[\overline{U}])| \geq 1$, either $|[U, \overline{U}]| \geq 4$ or both $G[U]$ and $G[\overline{U}]$ contain one of e_1 and e_2.

Proof of Claim 2. Suppose to the contrary that U is a smallest subset of $V(G)$ such that $E(G[U]) \neq \emptyset$, $|[U, \overline{U}]| \leq 3$, and $E(G[U]) \cap \{e_1, e_2\} = \emptyset$. Thus $G[U]$ is connected.

Let $G' = G/E(G[U])$, and u be the vertex resulting from a contraction of $E(G[U])$. Since (G, σ) is flow-admissible and $E(G[U]) \cap \{e_1, e_2\} = \emptyset$, (G', σ) is also flow-admissible, moreover, it admits a 6-NZF (τ', f') such that $f'(e_1) = f'(e_2) = 1$ by the minimality of $|E(G)|$.

Since $H(G') \subset H(G)$, let τ be an orientation of (G, σ) such that $\tau|_{H(G')} = \tau'$. Let G'' be the graph obtained from G by identifying all vertices of $\overline{U}$ as a single vertex, denoted by $\bar{u}$, and removing all resulting loops. Fix an orientation τ'' on $H(G'')$ as follows: for any $h \in H(G'')$,

$$\tau''(h) = \begin{cases} -\tau(h) & \text{if } h \in H_{G''}(\bar{u}) \text{ and the edge } e_h \text{ containing } h \text{ is in } \{e_1, e_2\}, \\ \tau(h) & \text{otherwise.} \end{cases}$$

Then τ'' can be viewed as an orientation on $E(G'')$ with all-positive edges by the assumption that $E(G[U]) \cap \{e_1, e_2\} = \emptyset$. By this assumption again, since (G, σ) is flow-admissible, G'' is 2-edge-connected and thus admits a 6-NZF by Theorem 1.1. Note that $d_{G''}(\bar{u}) = |[U, \overline{U}]| \leq 3$ and

$$\sum_{h \in H_{G''}(\bar{u})} \tau''(h)f'(e_h) = -\sum_{h \in H_{G'}(u)} \tau'(h)f'(e_h) = 0.$$

By Lemma 3.3, G'' admits a 6-NZF (τ'', f'') such that $f''(e) = f'(e)$ for each $e \in E_{G''}(\bar{u})$. Hence, (τ, f) is a desired 6-NZF on (G, σ), where $f(e) = f''(e)$ if $e \in E(G[U])$ and $f(e) = f'(e)$ otherwise. This contradicts that (G, σ) is a counterexample. □

Note that G contains no vertices of degree 2 by the minimality of $|E(G)|$. The following is an immediate corollary of Claim 2.

Claim 3. If T is an edge-cut of G with components Q_1, Q_2 such that Q_1 is all-positive, then $|T| \geq 3$.

Claim 4. There are two edge-disjoint circuits C_1 and C_2 of G such that $e_i \in E(C_i)$ for $i = 1, 2$.

Proof of Claim 4. Suppose not, since $\sigma^{-1}(-1) = \{e_1, e_2\}$, (G, σ) contains no two edge-disjoint unbalanced circuits. Since (G, σ) is flow-admissible, neither e_1 nor e_2 is a cut-edge of G, and so there exists a circuit of G containing e_1 and e_2. Further, G is 2-connected by the minimality of $|E(G)|$, and the frustration index is equal to 2. By Lemma 3.5 and Claim 1, there is a subset $S \subseteq E(G) \setminus \{e_1, e_2\}$ such that G/S is a connected cubic signed graph, which can be drawn in the plane with only one pair of crossing $\{e_1, e_2\}$. If $S \neq \emptyset$, then let B be a nontrivial component of $G[S]$. Thus $|E(B)| \geq 1$ and B is contracted into a vertex of G/S. Since G/S is cubic, $|E(G - V(B))| \geq 1$ and $|[V(B), \overline{V(B)}]_G| = 3$. By Claim 2, both B and $G - V(B)$ contain one of e_1 and e_2. This contradicts that $E(B) \subseteq S \subseteq E(G) \backslash \{e_1, e_2\}$. So $S = \emptyset$, and thus $G = G/S$.

Note that if a graph admits a k-NZF (D, h), then it admits an all-positive k-flow $(D', |h|)$, where D' is obtained from D by reversing the directions of all edges e with $h(e) < 0$. By Lemma 3.7, the underlying graph G is 3-edge-colorable, and so admits an all-positive 4-flow (τ', f') with $f'(e_1) = f'(e_2) = 1$ (the proof is referred to Exercise 3.14 in [23] or Lemma 20 in [12]).

Now, we are going to modify the 4-NZF (τ', f') on the underlying graph G to be a 6-NZF on the signed graph (G, σ). For $i = 1, 2$, let $e_i = u_i v_i$ and, without loss of generality, assume that u_i is oriented toward v_i under τ'. Let $G' = G - \{e_1, e_2\}$ and

$$U = \{u_1\} \cup \{u \in V(G) \ : \ G' \text{ contains a path } \mathcal{P}_{u_1 u} = x_1(=u_1)x_2 \cdots x_r(=u) \text{ such that } f'(x_i x_{i+1}) \neq 2 \text{ if } x_{i+1} \text{ is toward } x_i \text{ under } \tau'\}.$$

Suppose that $\overline{U} \neq \emptyset$. By the definition of U, every edge e in $[U, \overline{U}]_{G'}$ is oriented toward U under τ', moreover, $f'(e) = 2$ (see Figure 1). Since (τ', f') is a 4-NZF on G satisfying $f'(e) > 0$ for each $e \in E(G)$ and $f'(e_1) = f'(e_2) = 1$, $[U, \overline{U}]_{G'}$ consists of a unique edge, denoted by e_3, moreover, $[U, \overline{U}]_G = \{e_1, e_2, e_3\}$. Therefore, after switching at every vertex of U, the resulting signed graph obtained from (G, σ) contains a unique negative edge e_3, which contradicts that (G, σ) is flow-admissible. So $U = V(G)$.

Pick $\mathcal{P}_{u_1 v_2} = x_1(=u_1)x_2 \cdots x_r(=v_2)$ and let E_0 be the set of the edge $x_i x_{i+1}$ in $\mathcal{P}_{u_1 v_2}$ that is oriented toward x_i under τ'. Let h_1 (resp., h_2) be the half edge of $e_1 = u_1 v_1$ (resp., $e_2 = u_2 v_2$) incident with u_1 (resp., v_2), and use τ to denote the orientation of (G, σ) obtained from τ' by reversing the directions of h_1 and h_2. Then, we obtain a desired 6-NZF (τ, f) on (G, σ) with

$$f(e) = \begin{cases} f'(e) - 2 & \text{if } e \in E_0, \\ f'(e) + 2 & \text{if } e \in E(\mathcal{P}_{u_1 v_2}) \backslash E_0, \\ f'(e) & \text{otherwise.} \end{cases}$$

This contradicts that (G, σ) is a counterexample. □

By Claim 4, we can choose two edge-disjoint eulerian subgraphs H_1 and H_2 of G such that

(a) $e_i \in E(H_i)$ for $i = 1, 2$;
(b) subject to (a), the distance between H_1 and H_2 in G is as small as possible.

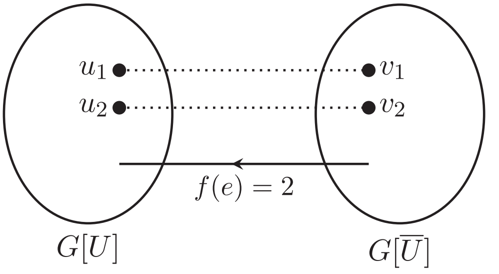

FIGURE 1 Any edge e in $[U, \overline{U}]_{G'}$

Let $P = x_1x_2\cdots x_t$ be a shortest path in G joining H_1 to H_2 such that $V(P) \cap V(H_1) = \{x_1\}$ and $V(P) \cap V(H_2) = \{x_t\}$. Then $t - 1$ is the distance between H_1 and H_2 in G. Note that P is a single vertex if $V(H_1) \cap V(H_2) \neq \emptyset$.

Claim 5. $t \leq 2$.

Proof of Claim 5. Suppose to the contrary that $t \geq 3$. Let $G' = G - \{x_1x_2, x_{t-1}x_t\}$.

We claim that G' contains a path joining $\{x_2, \ldots, x_{t-1}\}$ to $V(H_1) \cup V(H_2)$. Otherwise, $\{x_1x_2, x_{t-1}x_t\}$ is an edge-cut of G, and the component containing $\{x_2, \ldots, x_{t-1}\}$ in G' contains no negative edge. This contradicts Claim 3 since any such edge-cut must be of size at least 3. So the claim is true.

By the above claim, we pick a shortest path P_1 in G' joining $\{x_2, \ldots, x_{t-1}\}$ to $V(H_1) \cup V(H_2)$ (see Figure 2). Let x be the end of P_1 in $\{x_2, \ldots, x_{t-1}\}$ and, without loss of generality, assume that $y \in V(P_1) \cap (V(H_1) \cup V(H_2))$, say $y \in V(H_1)$. By (a), H_1 is an eulerian subgraph of G containing e_1, and so there is a trail, denoted by P_2, in H_1 connecting y with x_1 and containing e_1. Let $P(x_1, x)$ be the segment of P joining x_1 to x. Then $H'_1 = P(x_1, x) \cup P_1 \cup P_2$ is a new eulerian subgraph of G containing e_1. Moreover, $E(H'_1) \cap E(H_2) = \emptyset$ and the distance in G between H'_1 and H_2 is less than $t - 1$, a contradiction to (b). □

Let $H = H_1 \cup P \cup H_2$ and $H' = H - \{e_1, e_2\}$. Clearly, H' is a connected graph.

Claim 6. $G - E(H)$ contains a set of edge-disjoint circuits, say $\{C_1, \ldots, C_s\}$, such that

$$\langle H' \cup (\cup_{i=1}^{s} C_i) \rangle_2 = G - \{e_1, e_2\}.$$

Proof of Claim 6. By Claim 3, $(G - \{e_1, e_2\})/E(H') = G/E(H)$ is 3-edge-connected. So the claim follows from Lemma 3.2 □

The final step: Note that the subgraphs in $\{H_1 \cup P \cup H_2, C_1, \ldots, C_s\}$ are pairwise edge-disjoint by Claim 6. Since H_i $(i = 1, 2)$ is an eulerian subgraph of G with a unique negative edge e_i and P is a path joining H_1 to H_2, (G, σ) admits a 3-flow (τ, f_1) as follows. For $e \in E(G)$,

$$f_1(e) = \begin{cases} 1 & \text{if } e \in E(H_1) \cup E(H_2) \cup (\cup_{i=1}^{s} E(C_i)), \\ 2 & \text{if } e \in E(P) \text{ (if exists)}, \\ 0 & \text{otherwise.} \end{cases}$$

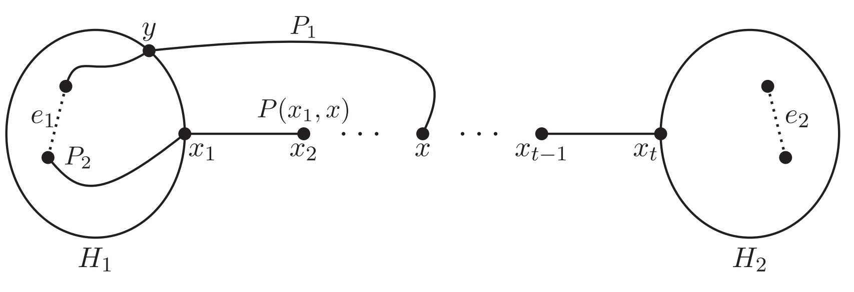

FIGURE 2 A shortest path P_1 joining $\{x_2, \ldots, x_{t-1}\}$ to $V(H_1) \cup V(H_2)$, and a trail P_2 joining y to x_1 and containing e_1

By Claim 6 and Lemma 3.1, G (and thus (G, σ)) admits a 3-flow (τ, f_2) such that $f_2(e_1) = f_2(e_2) = 0$ and

$$E(G)\setminus(E(H) \cup (\cup_{i=1}^{s} E(C_i))) \subseteq \text{supp}(f_2).$$

Note that either $E(P) = \emptyset$ or $E(P) = x_1x_2$ by Claim 5. Let

$$f = \begin{cases} f_1 - 2f_2 & \text{if } E(P) = x_1x_2 \text{ and } f_2(x_1x_2) \in \{-1, 2\}, \\ f_1 + 2f_2 & \text{otherwise.} \end{cases}$$

Then (τ, f) is a desired 6-NZF on (G, σ), which contradicts that (G, σ) is a counterexample. This completes the proof of Theorem 1.4 □

4 | PROOF OF THEOREM 1.5

For the sake of convenience, we use $\ell_0(G, \sigma)$ and $\ell(G, \sigma)$ to denote the frustration number and frustration index of a signed graph (G, σ), respectively.

Lemma 4.1. *Let λ and k be two given positive integers. Then, the following statements are equivalent.*

(1) *Every flow-admissible signed graph with frustration index at most λ admits a k-NZF.*
(2) *Every flow-admissible signed graph with frustration number at most λ admits a k-NZF.*

Proof.

(1) ==> (2) It is trivial since the frustration number is less than or equal to the frustration index in every signed graph.
(2) ==> (1) Suppose, to the contrary, that (2) is false. Let (G, σ) be a counterexample. Then $\ell_0(G, \sigma) \le \lambda$.

Let B be a subset of $V(G)$ with $|B| = \ell_0(G, \sigma)$ such that $(G - B, \sigma)$ is balanced. Note that switching does not change $\ell_0(G, \sigma)$ and $\ell(G, \sigma)$ in every signed graph (G, σ) (see Lemmas 7.1 and 7.2 in [22]). Then, we assume that all edges of $(G - B, \sigma)$ are positive.

We claim that the proof can be reduced to the case that B is an independent set. For each edge $e = uv \in E(G[B])$ (if exists), we replace e with a path uwv, and assign wu and wv two signatures as follows: both are positive if $\sigma(e) = 1$, and one is positive and the other is negative if $\sigma(e) = -1$. The resulting signed graph is denoted by (G', σ'). Since $E(G' - B) = E(G - B)$, all edges of $(G' - B, \sigma')$ are positive, and thus $\ell_0(G', \sigma') \le |B| \le \lambda$. By the structure of (G', σ'), (G', σ') admits an h-NZF if and only if so does (G, σ) for any positive integer h. So the claim follows from that $|E(G'[B])| < |E(G[B])|$.

Let X_u be the set of negative edges incident with u in (G, σ) for each $u \in B$. By the choice of B, $|X_u| \ge 1$ and $|E_G(u)\setminus X_u| \ge 1$. Since (G, σ) is flow-admissible, it admits an

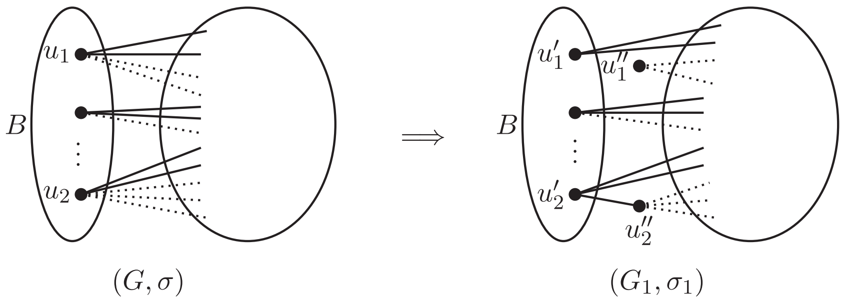

FIGURE 3 Construction of a new signed graph (G_1, σ_1) from (G, σ). Here, $\partial f(u_1'') = 0$ and $\partial f(u_2'') \neq 0$ under (τ, f). Positive edges are solid, and negative edges are dashed

NZF (τ, f). Thus, we construct a new signed graph (G_1, σ_1) from (G, σ) as follows (see Figure 3):

- for each $u \in B$ with $|X_u| \geq 2$, split the vertex u to a pair of vertices u' and u'', where u'' is incident with edges of X_u whereas u' is incident with the remaining edges of $E_G(u)$;
- add a new positive edge $u'u''$ if

$$\partial f(u'') = \sum_{h \in H(G) \text{ and } e_h \in X_u} \tau(h) f(e_h) \neq 0.$$

We prove that (G_1, σ_1) is flow-admissible. For every new edge $u'u''$, associate it with a direction from u' to u'' and assign it with flow value $\partial f(u'')$. Note that (τ, f) is an NZF on (G, σ). Then the resulting pair, denoted by (τ_1, f_1), obtained from (τ, f) is also an NZF on (G_1, σ_1), and thus (G_1, σ_1) is flow-admissible.

We prove that (G_1, σ_1) is of frustration index at most λ. For this aim, let σ_2 be the signature obtained from σ_1 by making a sequence of switchings on all new vertices u''. Note the assumptions that B is an independent set of G and all edges of $(G - B, \sigma)$ are positive. Then, every negative edge is incident with exactly one vertex of B in (G_1, σ_2). So $\ell(G_1, \sigma_2) \leq |B| \leq \lambda$, and thus $\ell(G_1, \sigma_1) = \ell(G_1, \sigma_2) \leq \lambda$ since switching does not change the frustration index.

By (1), (G_1, σ_1) admits a k-NZF. Then so does (G, σ), which contradicts that (G, σ) is a counterexample. $\square$

By applying Theorems 1.1 and 1.4, Theorem 1.5 is an immediate corollary of Lemma 4.1.

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