Received: 6 May 2021 | Accepted: 27 June 2021

DOI: 10.1002/jgt.22720

ARTICLE

WILEY

Berge–Fulkerson coloring for $C_{(12)}$-linked permutation graphs

Siyan Liu[1] | Rong-Xia Hao[1] | Cun-Quan Zhang[2] | Zhang Zhang[2,3]

[1]Department of Mathematics, Beijing Jiaotong University, Beijing, China

[2]Department of Mathematics, West Virginia University, Morgantown, West Virginia, USA

[3]Department of Mathematics, Beijing Institute of Technology, Beijing, China

Correspondence

Rong-Xia Hao, Department of Mathematics, Beijing Jiaotong University, Beijing, P.R. China.
Email: rxhao@bjtu.edu.cn

Cun-Quan Zhang, Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA.
Email: cqzhang@mail.wvu.edu

Funding information

National Science Foundation, Grant/Award Number: DMS-1700218; National Natural Science Foundation of China, Grant/Award Numbers: 11971054, 11731002

Abstract

It is conjectured by Berge and Fulkerson that *every bridgeless cubic graph has six perfect matchings such that each edge is contained in exactly two of them*. Let G be a permutation graph with a 2-factor $\mathcal{F} = \{C_1, C_2\}$. A circuit C_0 is $\mathcal{F}$-alternating if $E(C_0)\setminus(E(C_1) \cup E(C_2))$ is a perfect matching of C_0. A permutation graph G with a 2-factor $\mathcal{F} = \{C_1, C_2\}$ is $C_{(12)}$-linked if it contains an $\mathcal{F}$-alternating circuit of length at most 12. It is proved in this paper that *every* $C_{(12)}$*-linked permutation graph is Berge–Fulkerson colorable*. As an application, the conjecture is verified for some families of snarks constructed by Abreu et al., Brinkmann et al., and Hägglund et al.

KEYWORDS

Berge–Fulkerson coloring, Berge–Fulkerson conjecture, perfect matching, snark

1 | INTRODUCTION

The Berge–Fulkerson conjecture is one of the most famous open problems in graph theory. Although the statement of the Berge–Fulkerson conjecture is very simple, the solution has eluded many mathematicians over five decades and remains beyond the horizon.

Conjecture 1 (Berge–Fulkerson Conjecture [B-F-conjecture] [9], or see [15,16]). *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

A *snark* is a cyclically 4-edge connected cubic graph of girth at least 5 admitting no 3-edge coloring. The B-F-conjecture, similar to other major open problems, such as, Tutte's 5-flow conjecture, cycle double cover conjecture, is trivial for 3-edge-colorable cubic graphs, and remains widely open for snarks [17]. Among these famous conjectures, the B-F-conjecture is less explored than the other two conjectures and is still open for some known snarks. In [5,6,11,12,14], the conjecture is verified for some families of snarks. It was shown in [13], a possible minimum counterexample for the B-F-conjecture should have cyclic edge-connectivity at least 5.

The B-F-conjecture is equivalent to the statement that *every bridgeless cubic graph has a family of* six *cycles such that every edge is covered precisely* four *times*. It was proved by Bermond et al. [2] that every bridgeless graph has a family of seven cycles such that every edge is covered precisely four times; and Fan [7] proved that every bridgeless graph has a family of ten cycles such that every edge is covered precisely six times. The relation between the Berge–Fulkerson coloring and shortest cycle cover problems has been investigated by Fan and Raspaud [8].

Definition 1. Let G be a permutation graph.

(i) Let $\mathcal{F} = \{C_1, C_2\}$ be a chordless 2-factor. A circuit C_0 of G is *$\mathcal{F}$-alternating* if $E(C_0)\setminus(E(C_1) \cup E(C_2))$ is a perfect matching of C_0. This $\mathcal{F}$-alternating circuit C_0 is called a *linked circuit*.
(ii) The permutation graph G is *$C_{(\lambda)}$-linked* if it contains a chordless 2-factor $\mathcal{F}$ admitting an $\mathcal{F}$-alternating circuit of length at most λ.

The following is the main theorem of the paper.

Theorem 1. *Every $C_{(12)}$-linked permutation graph is Berge–Fulkerson colorable.*

In [12], the B-F-conjecture was verified for all permutation graphs with alternating circuit of length at most 8. This result is further extended to permutation graphs with alternating circuit of length 12 (Theorem 1). It is noticed that some families of snarks are $C_{(12)}$-linked but not $C_{(8)}$-linked. For example, twelve cyclically 5-edge connected permutation snarks discovered by Brinkmann, Goedgebeur, Hägglund, and Markström in [4], an infinite family of cyclically 5-edge connected permutation snarks discovered by Hägglund and Hoffmann-Ostenhof in [10]. As applications of the main result, the B-F-conjecture is further verified for these families of snarks.

In Section 2, some notations and definitions are presented. The proof of the main theorem (Theorem 1) is presented in Section 3. The applications are presented in Section 4. Further extensions and remarks are presented and discussed in the last section.

2 | PRELIMINARIES

Let $G = (V, E)$ be a graph. A *circuit* of G is a 2-regular connected subgraph. A *cycle* (or *an even graph*) is a graph with even degree at every vertex. The *suppressed graph*, denoted by $\overline{G}$, is the graph obtained from G by suppressing all degree-2-vertices. A *k-factor* of a graph G is a spanning k-regular subgraph of G. The set of edges of a 1-factor of a graph G is called a *perfect matching* of G. We refer to [3] for notation and terminologies used but not defined here.

Let X and Y be two subgraphs of G. The *symmetric difference* of X and Y, denoted by $X \triangle Y$, is the subgraph of G induced by the edge set $(E(X) \cup E(Y))\setminus(E(X) \cap E(Y))$. The set $\{1, 2, ..., n\}$ is denoted by $[n]$.

A cubic graph G is *Berge–Fulkerson colorable* if $2G$ is 6-edge-colorable, where the graph $2G$ is obtained from G by replacing every edge with a pair of parallel edges. It is obvious that this is an equivalent description of the B-F-conjecture.

Lemma 1 (Hao et al. [11]). *A cubic graph G is Berge–Fulkerson colorable if and only if there are two edge-disjoint matchings M_1 and M_2 such that*

(1) *$M_1 \cup M_2$ is an even subgraph Q in G, and*
(2) *for each $i = 1, 2$ and for each component X of $G \backslash M_i$, either the suppressed graph $\overline{X}$ is 3-edge-colorable, or, X is a circuit.*

An equivalent statement of Lemma 1 for cubic graphs can be found in [6].

The following observation (Proposition 1) ensures the existence of an $\mathcal{F}$-alternating circuit in any permutation graph.

Proposition 1. *Every permutation graph with a chordless 2-factor $\mathcal{F} = \{C_1, C_2\}$ has an $\mathcal{F}$-alternating circuit C_0 of length $4k$, where $|V(C_1)| = |V(C_2)| \geq 3$.*

Proof. Let G be a given permutation graph with a chordless 2-factor $\mathcal{F} = \{C_1, C_2\}$ and a perfect matching $M = E(G) \backslash (E(C_1) \cup E(C_2))$, where $|V(C_1)| = |V(C_2)| = n \geq 3$. Let $C_1 = v_1 \cdots v_n v_1$ and $C_2 = u_1 \cdots u_n u_1$. Without loss of generality, suppose v_n is adjacent to u_n.

Let $\widehat{G} = G$ if $n = 2t$ is even, or, $\widehat{G} = \overline{G - \{v_n u_n\}}$ if $n = 2t + 1$ is odd. Let $\widehat{\mathcal{F}} = \{\hat{C}_1, \hat{C}_2\}$ be the corresponding chordless 2-factor of $\widehat{G}$, where $\hat{C}_1 = v_1 \cdots v_{2t} v_1$ and $\hat{C}_2 = u_1 \cdots u_{2t} u_1$.

Assign a 3-edge-coloring mapping $\sigma: E(\hat{G}) \to \{\text{Red, Blue, Yellow}\}$ such that

(1) the edges in $\hat{C}_1$ and $\hat{C}_2$ are alternately colored *Red* and *Blue* with $u_{2t} u_1$ and $v_{2t} v_1$ colored *Red*, and
(2) the edges in $E(\widehat{G}) \backslash (E(\hat{C}_1) \cup E(\hat{C}_2))$ are colored with *Yellow*.

Any *Blue-Yellow* bicolored circuit is an $\mathcal{F}$-alternating circuit of length $4k$ in G. □

3 | THE PROOF OF THEOREM 1

Let G be a counterexample to the theorem with a chordless 2-factor $\mathcal{F} = \{C_1, C_2\}$ and a perfect matching $M = E(G) \backslash (E(C_1) \cup E(C_2))$, where

$$C_1 = v_1 \cdots v_n v_1, \quad C_2 = u_1 \cdots u_n u_1.$$

Assume that n is odd, otherwise the graph G is 3-edge-colorable.

Let $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation on the set $\{1, \ldots, n\}$ such that

$$M = \{v_i u_{\pi(i)}: \ i \in [n]\}.$$

Let C_0 be an $\mathcal{F}$-alternating circuit of length at most 12.

Since it was proved in [12] that Conjecture 1 holds for the case of $C_{(8)}$-linked permutation graphs, we have the following claim (by Proposition 1).

Claim 1. C_0 is of length 12, and G does not have any $\mathcal{F}$-alternating circuit of length 4 or 8.

Let $J = C_0 \triangle C_1 \triangle C_2$, and let $\{J_1, ..., J_\alpha\}$ be the set of all components of J, where α is the number of components of J and

$$|E(J_1) \cap E(C_0)| \leq |E(J_2) \cap E(C_0)| \leq \cdots \leq |E(J_\alpha) \cap E(C_0)|.$$

Claim 2. α is either 2 or 3.

Proof of Claim 2. Since $|E(C_0) \cap M| = 6$ and there is an even number of edges of $E(C_0) \cap M$ in each component J_i of J, α must be at most 3.

If $\alpha = 1$, then J is a Hamilton circuit of G and, therefore, G is 3-edge-colorable. This contradicts that G is a counterexample, and therefore, the claim is proved. □

When $\alpha = 2$, J has two components,

$$|E(J_1) \cap E(C_0)| = 2 \quad \text{and} \quad |E(J_2) \cap E(C_0)| = 4.$$

When $\alpha = 3$, J has three components, and, similarly,

$$|E(J_i) \cap E(C_0)| = 2, \quad \forall\, i = 1, 2, 3.$$

Claim 3. Two components of J, say J_β and J_γ, are of odd orders ($\beta, \gamma \in [\alpha]$).

Proof of Claim 3. Note that $J = C_0 \triangle C_1 \triangle C_2$ is a 2-factor of G. If all of its components are of even order, then G is 3-edge-colorable, a contradiction. Since the number of odd components of J must be even, by Claim 2, J has precisely two odd components. Without loss of generality, let them be J_β and J_γ. □

Notation. Let $E(C_0) \cap E(C_1) = \{v_t v_{t+1}, v_s v_{s+1}, v_k v_{k+1}\}$, and denote

$$E(C_0) \cap M = \{v_\mu u_{\pi(\mu)} :\ \mu = t, t+1, s, s+1, k, k+1\}.$$

Let L_1, L_2, and L_3 be the components of $E(C_1) \backslash E(C_0)$, each of which is a path with end vertices v_t, v_{k+1}, v_k, v_{s+1}, and v_s, v_{t+1}, respectively, as can be seen in Figure 1.

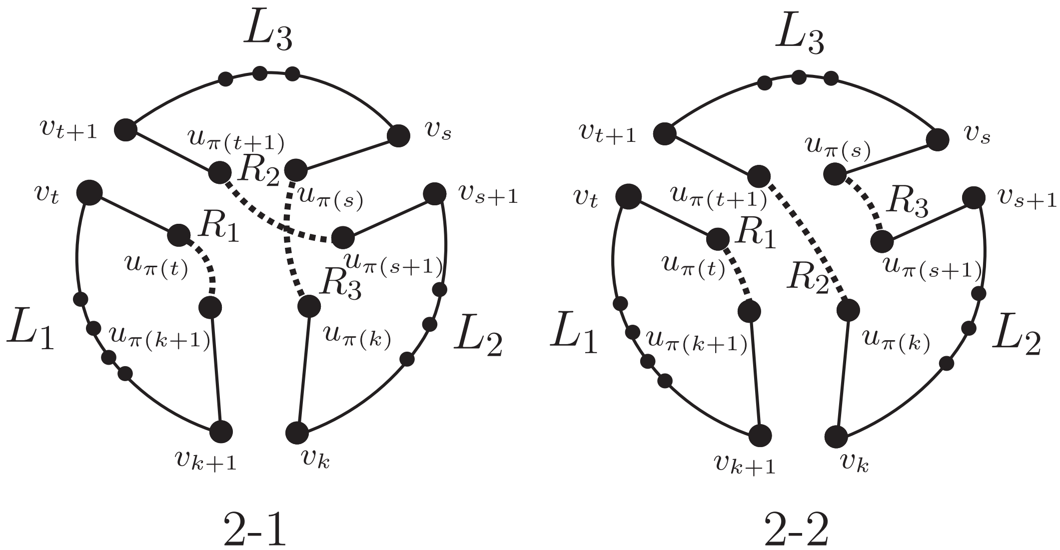

FIGURE 1 The types of connections when $\alpha = 2$, with R_1, R_2, R_3 shown as dotted lines

Let $X = V(C_0) \cap V(C_2)$. The circuit C_2 is the union of six segments of C_2 separated by X, in which three of them belonging to C_0 are single edges, and the other three paths are components of $E(C_2)\backslash E(C_0)$. Denote them by e_1, e_2 , e_3, R_1, R_2, and R_3, respectively.

Without loss of generality, let L_1 of C_1, R_1 of C_2 be contained in J_β (together with two edges of C_0). That is,

$$J_\beta = L_1 \cup R_1 \cup \{v_{k+1}u_{\pi(k+1)}, v_t u_{\pi(t)}\}.$$

By Claim 3, the circuit J_β is of odd length. That is, the lengths of L_1 and R_1 are of different parity. Without loss of generality, we assume that

Assumption. *L_1 is of odd length, and R_1 is of even length.*

(Note that if L_1 is of even length, and R_1 is of odd length. One may interchange C_1 and C_2).

With all these claims and the above assumption, we are ready to find all possible configurations (up to isomorphism) in the next two claims.

Claim 4. If $\alpha = 2$, there are five configurations T_j for $j \in [5]$ (see Figure 2).

Proof of Claim 4. As $\alpha = 2$, we notice that, up to isomorphism, $\{R_2, R_3\}$ has precisely two types of connections, see Figure 1. Call them Type 2-1 (the left one), and Type 2-2 (the right one).

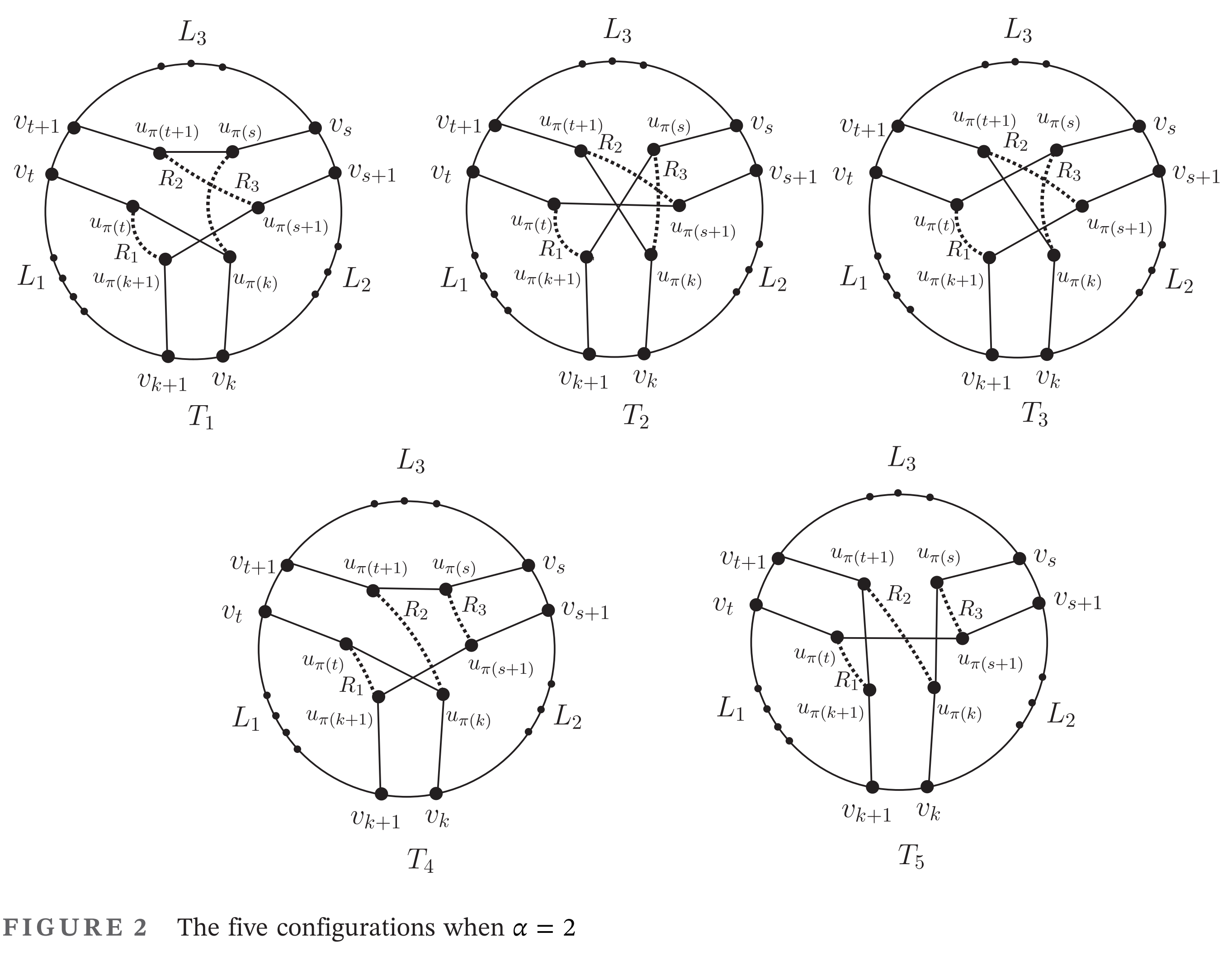

FIGURE 2 The five configurations when $\alpha = 2$

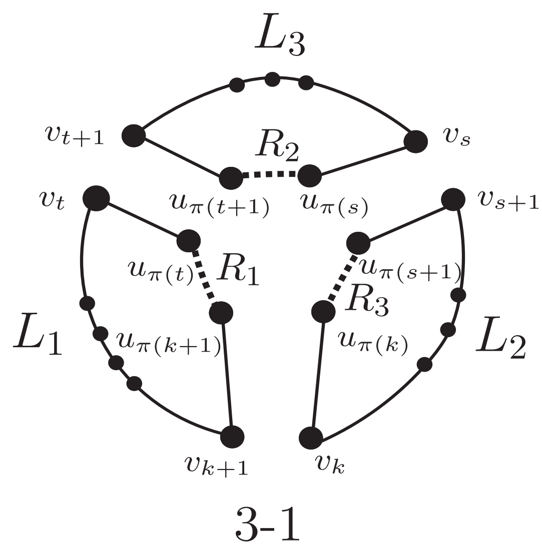

FIGURE 3 The types of connections when $\alpha = 3$, with R_1, R_2, R_3 shown as dotted lines

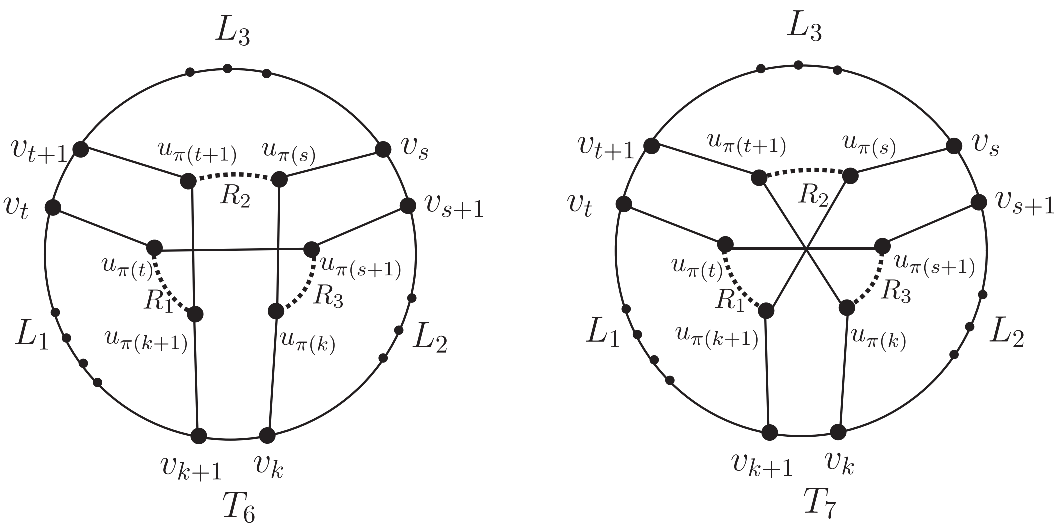

FIGURE 4 The two configurations when $\alpha = 3$

Furthermore, for Type 2-1, by applying Claim 1 (avoiding $\mathcal{F}$-alternating 4-circuit and 8-circuit), edges e_1, e_2, and e_3 have precisely three different distributions, denoted by T_1, T_2, and T_3. For Type 2-2, edges e_1, e_2, and e_3 are distributed in two ways, denoted by T_4 and T_5 (see Figure 2). □

Claim 5. If $\alpha = 3$, there are two configurations T_6 and T_7 (see Figure 4).

Proof of Claim 5. As $\alpha = 3$, we notice that, up to isomorphism, the connection of each of R_2 and R_3 is uniquely determined, call it Type 3-1 (see Figure 3). Then, by applying Claim 1 (avoiding $\mathcal{F}$-alternating 4-circuit and 8-circuit), edges e_1, e_2, and e_3 have precisely two different distributions, denoted by T_6 and T_7 (see Figure 4). □

Since C_1 is of odd length, the total length of L_1, L_2. and L_3 is even. Moreover, since L_1 is of odd length in the odd component J_β (by the Assumption), we have the following obvious claim.

Claim 6. The lengths of L_2 and L_3 are of different parity.

By Claim 6, there are two cases. (Now, we are ready to apply Lemma 1 by finding the circuit Q).

Case I. If L_2 is of even length and L_3 is of odd length.
Let

$$Q = \begin{cases} v_{s+1}L_2v_kv_{k+1}u_{\pi(k+1)}u_{\pi(s+1)}v_{s+1}, & \text{if } G \text{ is one of } T_1 \text{ and } T_4; \\ v_tv_{t+1}u_{\pi(t+1)}u_{\pi(k)}v_kv_{k+1}L_1v_t, & \text{if } G \text{ is one of } T_2, T_3 \text{ and } T_7; \\ v_tv_{t+1}L_3v_sv_{s+1}u_{\pi(s+1)}u_{\pi(t)}v_t, & \text{if } G \text{ is one of } T_5 \text{ and } T_6. \end{cases} \tag{1}$$

Case II. If L_2 is of odd length and L_3 is of even length.
Let

$$Q = \begin{cases} v_tv_{t+1}u_{\pi(t+1)}u_{\pi(s)}v_sv_{s+1}L_2v_kv_{k+1}L_1v_t, & \text{if } G \text{ is one of } T_1 \text{ and } T_4; \\ v_tv_{t+1}u_{\pi(t+1)}u_{\pi(k)}v_kv_{k+1}L_1v_t, & \text{if } G \text{ is one of } T_2, T_3 \text{ and } T_7; \\ v_{t+1}L_3v_sv_{s+1}u_{\pi(s+1)}u_{\pi(t)}v_tL_1v_{k+1}u_{\pi(k+1)}u_{\pi(t+1)}v_{t+1}, & \text{if } G \text{ is one of } T_5 \text{ and } T_6. \end{cases} \tag{2}$$

Let M_1, M_2 be a pair of edge-disjoint perfect matchings of Q. Without loss of generality, let $E(Q)\setminus(E(C_1) \cup E(C_2)) \subseteq M_2$.

Let

$$N_1 = Q\triangle(C_1 \cup C_2), \quad \text{and} \quad N_2 = C_0\triangle N_1.$$

Note that each N_i is a Hamilton circuit in $\overline{G\setminus M_i}$ for each $i \in [2]$.

We deal with the configuration T_1 in Case I as an example. The Hamilton circuit N_i in $\overline{G\setminus M_i}$ is highlighted as bold lines/curves in Figure 5.

Consequently, the suppressed cubic graph $\overline{G\setminus M_i}$ is 3-edge-colorable for every configuration T_j ($j \in [7]$) and each matching M_i ($i \in [2]$). By Lemma 1, the graph G admits a Berge–Fulkerson coloring, a contradiction. Therefore every $C_{(12)}$-linked permutation graph is Berge–Fulkerson colorable.

This completes the proof of Theorem 1.

4 | APPLICATIONS: BERGE–FULKERSON COLORINGS OF SOME FAMILIES OF SNARKS

In this section, the conjecture is verified for some infinite families of snarks.

4.1 | Hägglund–Hoffmann-Ostenhof (HHO) snarks

In [10], an infinite family of cyclically 5-edge connected permutation snarks, denoted by *HHO*-snarks, was presented by Hägglund and Hoffmann-Ostenhof. As a corollary of the main result in this paper, the B-F-conjecture is verified for *HHO*-snarks.

Notation. Denote by P_{10} the Pertersen graph.

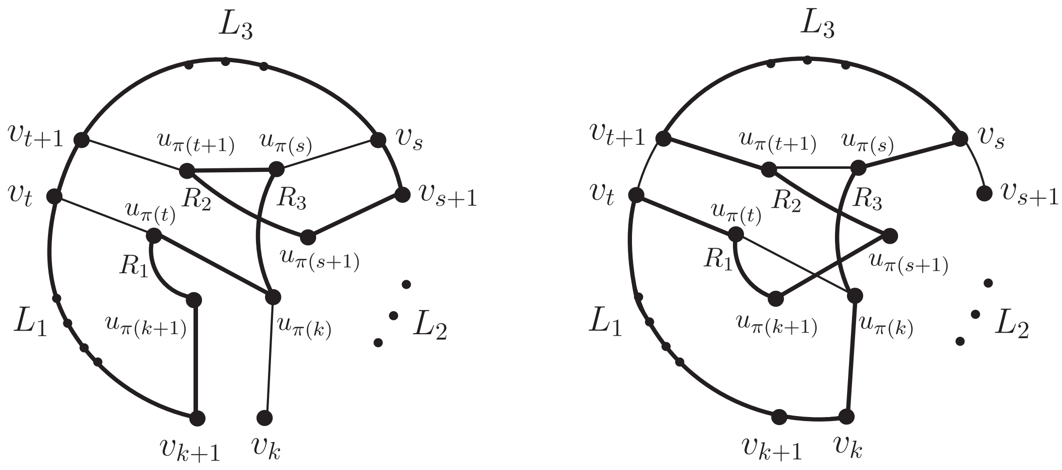

FIGURE 5 Illustration of Hamilton circuits N_i in $\overline{G\backslash M_i}$ of T_1 for Case I

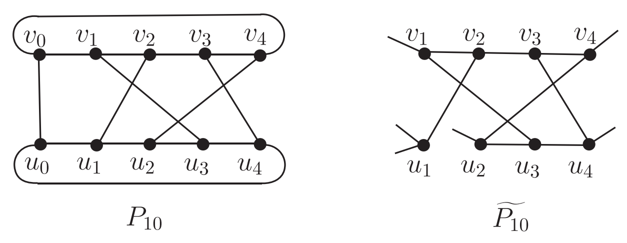

FIGURE 6 Illustration of P_{10} and $\widetilde{P_{10}}$

The edges which are associated with just one vertex are called *semiedges*. Denote by $\tilde{u}$ the semiedge which is adjacent to vertex u. A *multipole* is a graph in which semiedges are allowed. Define a *join* between two semiedges $\tilde{u}$ and $\tilde{v}$ as the removal of semiedges $\tilde{u}$ and $\tilde{v}$, and the addition of edge uv.

Definition 2. Let G be a permutation graph with a chordless 2-factor $\mathcal{F} = \{C_1, C_2\}$, where $C_1 = v_0v_1v_2\cdots v_nv_0$ and $C_2 = u_0u_1u_2\cdots u_nu_0$. Let v_0u_0 be an edge of G with $v_0 \in C_1$ and $u_0 \in C_2$. Note that v_1, v_n and u_1, u_n be the other neighbours of v_0 and u_0, respectively. Let u_2 be the neighbour in C_2 of u_1 and $u_2 \neq u_0$. Let $\widetilde{G}$ be the graph obtained from G by removing vertices v_0 and u_0 and the edge u_1u_2, and adding one semiedge to vertices v_1, v_n, u_2 and u_n, and adding two semiedges to vertex u_1. We shall refer to $\widetilde{G}$ as the multipole of G with respect to the edge v_0u_0.

Let P_{10} be a Petersen graph with a chordless 2-factor $\mathcal{F} = \{C_1, C_2\}$, where $C_1 = v_0v_1v_2v_3v_4v_0$ and $C_2 = u_0u_1u_2u_3u_4u_0$. The graph P_{10} and the multipole $\widetilde{P_{10}}$ with respect to the edge v_0u_0 are shown in Figure 6.

The permutation graphs H_i, for $i \geq 1$, are given in [10] and are constructed recursively as follows.

Let H_1 be the graph obtained from four copies of multipole $\widetilde{P_{10}}$ and from two new adjacent vertices p_1 and q_1 by joining semiedges of the multipoles and the vertices p_1 and q_1 to the rest of the graph as in Figure 7. H_1 is a permutation graph with a chordless 2-factor $\mathcal{F}_1 = \{C_1^1, C_1^2\}$, where $C_1^1 = v_3v_4y_2y_3y_4n_4n_3n_2a_4a_3a_2a_1n_1q_1y_1v_1v_2v_3$ and $C_1^2 = H_1\backslash V(C_1^1)$. In fact $q_1 \in C_1^1$, $\quad p_1 \in C_1^2$.

The graph H_1, shown in Figure 7, is denoted by $H(P_{10}, P_{10}, P_{10}, P_{10})$.

H_n is recursively constructed as follows.

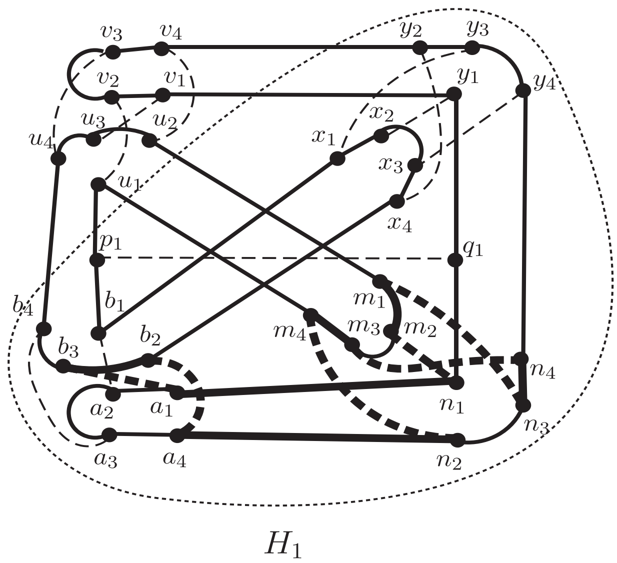

FIGURE 7 Illustration of the graph H_1

For a given H_{n-1} with a chordless 2-factor $\mathcal{F}_{n-1} = \{C_{n-1}^1, C_{n-1}^2\}$, let $\widetilde{H_{n-1}}$ be the multipole of H_{n-1} (see Definition 2). H_n is constructed from $\widetilde{H_{n-1}}$, three copies of $\widetilde{P_{10}}$ and two new adjacent vertices p_n, q_n by joining semiedges in a similar manner as in Figure 7, and the vertices p_n, q_n to the rest of the graph, which is denoted by $H(H_{n-1}, P_{10}, P_{10}, P_{10})$ and is shown in Figure 8. The structure of H_n is exactly the same as H_1, except that one copy of P_{10} is replaced by H_{n-1}. The details are referred to [10].

Lemma 2 (Hägglund and Hoffmann-Ostenhof [10]). *Let $\mathcal{H} := \bigcup_{n=0}^{\infty}\{H_n\}$ be the HHO-snarks family with $H_0 := P_{10}$ and for $n \geq 1$, $H_n := H(H_{n-1}, P_{10}, P_{10}, P_{10})$. Then, $\mathcal{H}$ is an infinite family of cyclically 5-edge connected permutation snarks, where $H_n \in \mathcal{H}$ has $10 + 24n$ vertices.*

According to the construction of H_n, each H_i for $i \in [n]$ has the same structure which is shown inside the dotted line in Figures 7 and 8. This same structure is also shown in Figure 9 which is the local structure of H_n with a $C_{(12)}$-linked circuit $C_{(12)} = m_1m_2n_1a_1b_3b_2a_4n_2m_4m_3n_4n_3m_1$ between the 2-factor $\mathcal{F} = \{C_1, C_2\}$ of the permutation graph H_n. Thus, each H_n for $i \in [n]$ is a $C_{(12)}$-linked permutation graph.

As a corollary of Theorem 1, every member of the infinite family snarks $\mathcal{H}$ is Berge–Fulkerson colorable. That is as follows.

Corollary 1. *Every H_n in the infinite set of cyclically 5-edge connected permutation snarks $\mathcal{H} := \bigcup_{n=0}^{\infty}\{H_n\}$ is Berge–Fulkerson colorable.*

4.2 | Brinkmann–Goedgebeur–Hägglund–Markström snarks

In [4], 12 cyclically 5-edge connected permutation snarks on 34 vertices have been discovered by Brinkmann, Goedgebeur, Hägglund, and Markström using a computer search and denoted them by $BGHM_{34}$-snarks. The B-F-conjecture was also verified for all of them in [4] by finding a Petersen coloring. Note that finding a Petersen coloring is not a straightforward process. In this

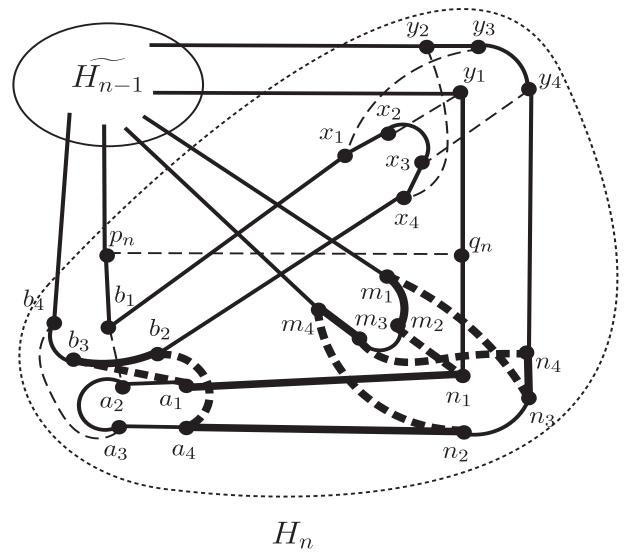

FIGURE 8 Illustration of the graph H_n

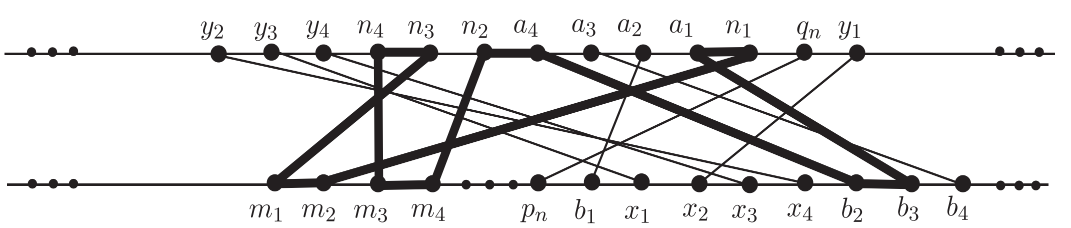

FIGURE 9 A $C_{(12)}$-alternating circuit in H_n

paper, as a different approach, the B-F-conjecture is verified again for those snarks by applying our main theorem, as each of them is a $C_{(12)}$-linked permutation graph.

Let G_i be a $BGHM_{34}$-snark ($i \in [12]$). A chordless 2-factor $\mathcal{F}_i = \{C_i^1, C_i^2\}$ of G_i is listed as follows:

$C_1^1 = v_{25}v_{26}v_{19}v_{20}v_{30}v_{32}v_{18}v_{22}v_{14}v_{11}v_7v_{23}v_8v_4v_{16}v_6v_2v_{25}$,
$C_1^2 = v_1v_9v_{10}v_{12}v_3v_{31}v_{21}v_5v_{27}v_{28}v_{33}v_{34}v_{24}v_{13}v_{29}v_{15}v_{17}v_1$.
$C_2^1 = v_9v_{30}v_{10}v_{12}v_{16}v_{11}v_7v_{20}v_{34}v_{26}v_{22}v_{21}v_{28}v_5v_{31}v_6v_4v_9$,
$C_2^2 = v_1v_{13}v_{14}v_2v_{15}v_{24}v_{32}v_{23}v_{25}v_8v_{33}v_3v_{29}v_{18}v_{17}v_{19}v_{27}v_1$.
$C_3^1 = v_{12}v_{16}v_{22}v_{21}v_{14}v_{13}v_{27}v_{31}v_{32}v_{24}v_{20}v_{18}v_{15}v_6v_4v_8v_3v_{12}$,
$C_3^2 = v_1v_9v_{23}v_{10}v_{29}v_{30}v_{33}v_{34}v_{19}v_5v_{26}v_{28}v_{25}v_2v_7v_{11}v_{17}v_1$.
$C_4^1 = v_9v_{10}v_{12}v_{22}v_{16}v_{11}v_7v_{19}v_{20}v_{18}v_{17}v_{33}v_{34}v_5v_{31}v_6v_4v_9$,
$C_4^2 = v_1v_{21}v_{30}v_{29}v_3v_{27}v_{28}v_8v_{25}v_{32}v_{26}v_{15}v_2v_{14}v_{13}v_{24}v_{23}v_1$.
$C_5^1 = v_{22}v_{21}v_{31}v_5v_{19}v_{28}v_{27}v_{33}v_{34}v_{24}v_{26}v_{30}v_{25}v_{11}v_7v_8v_4v_{22}$,
$C_5^2 = v_1v_9v_{17}v_{20}v_{18}v_{14}v_{29}v_{32}v_{13}v_6v_{23}v_2v_{10}v_{12}v_3v_{16}v_{15}v_1$.
$C_6^1 = v_{19}v_{23}v_{24}v_{11}v_{22}v_{21}v_{30}v_3v_5v_{32}v_{31}v_{27}v_{28}v_{25}v_6v_{13}v_2v_{19}$,
$C_6^2 = v_1v_{15}v_{16}v_{14}v_{12}v_{26}v_{10}v_9v_4v_8v_7v_{29}v_{33}v_{34}v_{20}v_{18}v_{17}v_1$.
$C_7^1 = v_{12}v_{24}v_{14}v_{13}v_{25}v_{30}v_{26}v_{22}v_{33}v_{34}v_{32}v_{31}v_{19}v_{17}v_{28}v_{18}v_{10}v_{12}$,
$C_7^2 = v_1v_9v_{27}v_{29}v_4v_8v_{20}v_{16}v_3v_5v_6v_{23}v_2v_7v_{11}v_{15}v_{21}v_1$.
$C_8^1 = v_{32}v_{10}v_{26}v_{12}v_{14}v_{16}v_{22}v_{15}v_5v_6v_{19}v_{20}v_{18}v_8v_{34}v_{33}v_2v_{32}$,
$C_8^2 = v_1v_9v_4v_{27}v_{28}v_{13}v_{30}v_{29}v_{21}v_3v_{24}v_{23}v_7v_{11}v_{17}v_{25}v_{31}v_1$.
$C_9^1 = v_9v_{10}v_{16}v_{15}v_5v_{18}v_{20}v_{21}v_{31}v_{32}v_{27}v_{11}v_7v_{24}v_{23}v_6v_4v_9$,
$C_9^2 = v_1v_{19}v_{34}v_{33}v_{29}v_{30}v_8v_3v_{12}v_{14}v_{22}v_{13}v_2v_{17}v_{26}v_{28}v_{25}v_1$.
$C_{10}^1 = v_{27}v_{28}v_{11}v_{14}v_{13}v_{32}v_{23}v_4v_{17}v_9v_{10}v_{22}v_{21}v_{19}v_5v_{29}v_3v_{27}$,
$C_{10}^2 = v_1v_{15}v_{16}v_8v_7v_2v_6v_{25}v_{26}v_{30}v_{20}v_{24}v_{18}v_{31}v_{12}v_{34}v_{33}v_1$.

$$C_{11}^1 = v_{29}v_8v_{23}v_4v_9v_{10}v_{25}v_{28}v_{26}v_{22}v_{34}v_{33}v_{14}v_{32}v_{18}v_{17}v_3v_{29},$$
$$C_{11}^2 = v_1v_{11}v_{21}v_7v_2v_{13}v_6v_{31}v_5v_{15}v_{16}v_{27}v_{12}v_{30}v_{24}v_{20}v_{19}v_1.$$
$$C_{12}^1 = v_{31}v_{21}v_{22}v_{17}v_{28}v_{27}v_{14}v_{30}v_{29}v_{16}v_{24}v_{23}v_8v_4v_6v_5v_3v_{31},$$
$$C_{12}^2 = v_1v_9v_{34}v_{33}v_7v_{32}v_{11}v_{12}v_{10}v_2v_{25}v_{26}v_{15}v_{18}v_{20}v_{19}v_{13}v_1.$$

Lemma 3. *The twelve $BGHM_{34}$-snarks are $C_{(12)}$-linked permutation graphs.*

Proof. Let G_i be a $BGHM_{34}$-snark ($i \in [12]$), and $\mathcal{F}_i$ be described as above. An $\mathcal{F}_i$-alternating circuit C_i^0 in G_i is listed as follows for each i. It can be checked that each C_i^0 is a $C_{(12)}$-linked circuit, as required.

$$C_1^0 = v_{19}v_{26}v_5v_{27}v_6v_{16}v_{15}v_{17}v_{20}v_{30}v_{29}v_{13}v_{19}.$$
$$C_2^0 = v_{12}v_{10}v_{14}v_{13}v_{21}v_{22}v_{19}v_{17}v_9v_{30}v_{29}v_{18}v_{12}.$$
$$C_3^0 = v_{14}v_{21}v_{10}v_{23}v_{32}v_{24}v_{17}v_1v_{31}v_{27}v_{28}v_{25}v_{14}.$$
$$C_4^0 = v_{12}v_{22}v_{21}v_{30}v_9v_4v_{25}v_{32}v_{31}v_5v_3v_{29}v_{12}.$$
$$C_5^0 = v_{33}v_{34}v_{23}v_2v_7v_{11}v_{14}v_{18}v_{26}v_{24}v_{12}v_{10}v_{33}.$$
$$C_6^0 = v_{22}v_{21}v_{12}v_{26}v_{28}v_{25}v_4v_8v_3v_{30}v_{29}v_7v_{22}.$$
$$C_7^0 = v_{14}v_{13}v_4v_8v_{26}v_{22}v_{21}v_{15}v_{34}v_{33}v_7v_{11}v_{14}.$$
$$C_8^0 = v_{26}v_{12}v_{29}v_{21}v_{22}v_{16}v_{30}v_{13}v_2v_{32}v_{31}v_{25}v_{26}.$$
$$C_9^0 = v_{16}v_{15}v_{19}v_1v_{11}v_7v_{26}v_{28}v_{27}v_{32}v_{14}v_{12}v_{16}.$$
$$C_{10}^0 = v_{11}v_{14}v_{18}v_{24}v_{23}v_{32}v_{31}v_{12}v_3v_{27}v_{16}v_{15}v_{11}.$$
$$C_{11}^0 = v_8v_{23}v_{24}v_{30}v_{29}v_3v_{20}v_{19}v_{34}v_{22}v_{21}v_7v_8.$$
$$C_{12}^0 = v_{22}v_{17}v_1v_{13}v_{27}v_{28}v_{20}v_{19}v_5v_3v_{12}v_{11}v_{22}.$$

As an example, the snark G_1 is shown in Figure 10, in which the 2-factor $\mathcal{F}_1 = \{C_1^1, C_1^2\}$ of G_1 is shown inside the dotted line. □

As a consequence of Theorem 1, we have the following corollary.

Corollary 2. *The 12 $BGHM_{34}$-snarks are Berge–Fulkerson colorable.*

4.3 | Abreu–Labbate–Rizzi--heehan snark

In this section, a snark of order 26, denoted by $ALRS_{26}$ and discovered in [1] (see Figure 11) is verified for the B-F-conjecture.

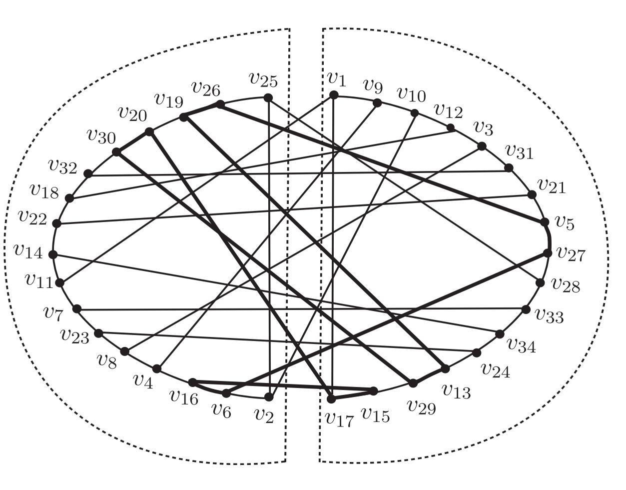

FIGURE 10 Illustration of the $BGHM_{34}$-snark G_1

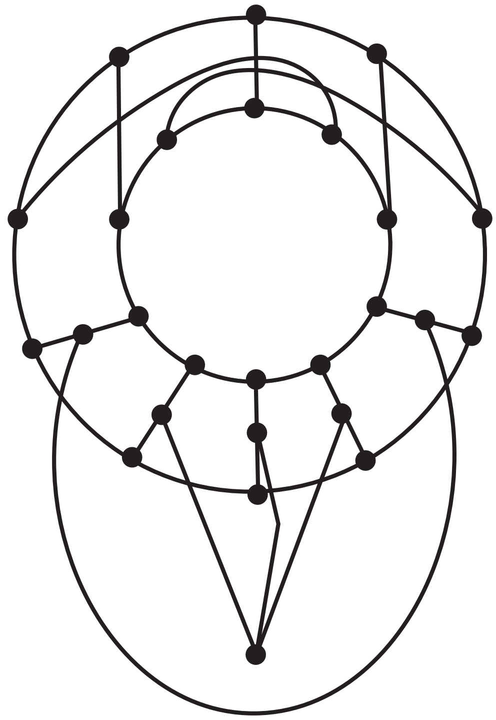

FIGURE 11 Illustration of the snark $ALRS_{26}$

Actually, due to a special substructure of the $ALRS_{26}$-snark, we are able to verify the B-F-conjecture for a larger family of cubic graphs, and the $ALRS_{26}$-snark is a member of this family.

Definition 3. Let G' be a permutation graph with a 2-factor $\mathcal{F} = \{C_1, C_2\}$ and let $M = E(G')\backslash(E(C_1) \cup E(C_2))$ be the perfect matching between C_1 and C_2 such that $v_1v_2\cdots v_5$ and $u_1u_2\cdots u_5$ are paths in C_1 and C_2, respectively, and $v_iu_i \in M$ for $i \in [5]$. Let G be a graph obtained from G' by inserting one vertex, say w_i, in the edge v_iu_i for each $i \in [5]$, a new vertex w and edges w_1w_5, w_iw for $i = 2, 3, 4$ (see Figure 12).

Every graph G defined in Definition 3 is called *an ALRS-cubic graph*.

Corollary 3. *Every ALRS-cubic graph G is Berge–Fulkerson colorable.*

Proof. Let M_1 and M_2 be two distinct matchings such that $M_1 \bigcup M_2$ is the 6-circuit $v_2v_3v_4w_4ww_2v_2$ (see Figure 12). Then, each of $\overline{G\backslash M_1}$ and $\overline{G\backslash M_2}$ contains a Hamilton circuit and so is 3-edge-colorable. By Lemma 1, the graph G is Berge-Fulkerson colorable. □

5 | EXTENSIONS AND REMARKS

The notion of an $\mathcal{F}$-alternating circuit in Definition 1 was defined for *permutation graphs*, in which each component of a 2-factor is chordless. However, we note that the existence of chord in a 2-factor has little impact on the proof of Theorem 1. Thus, we may modify Definition 1 and extend Theorem 1 as follows.

Definition 4. Let $\mathcal{F}$ be a 2-factor of a cubic graph G with the set of components $\{C_1, C_2\}$. A circuit $C_0 = e_1e_2\cdots e_{2t}$ is $\mathcal{F}$*-alternating* if, for every $i \in [t]$, e_{2i} is an edge of the

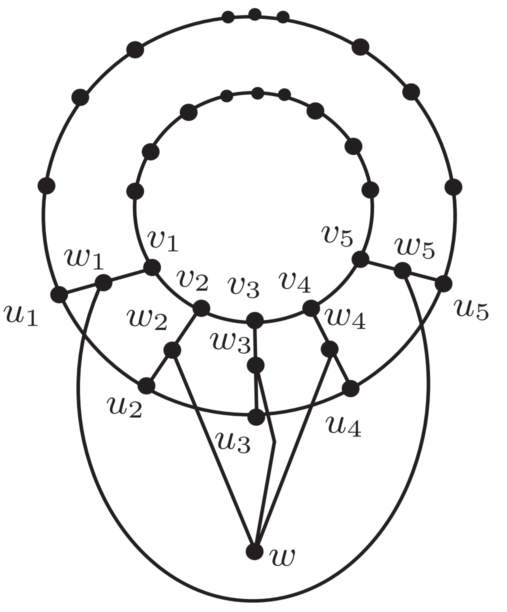

FIGURE 12 Illustration of *ALRS*-cubic graphs

matching $M \subseteq E(G) \backslash (E(C_1) \cup E(C_2))$ joining C_1 and C_2, but *not a chord of any component* of $\mathcal{F}$.

With Definition 4, Theorem 1 is extended as follows.

Theorem 2. *Let $\mathcal{F} = \{C_1, C_2\}$ be a 2-factor of a cubic graph G. If G contains an $\mathcal{F}$-alternating circuit of length at most 12, then G is Berge-Fulkerson colorable.*

Note that the proof of Theorem 1 is based on the subgraph of G induced by the edges of C_1, C_2, and an $\mathcal{F}$-alternating circuit C_0. Therefore, the condition of chordless is not used at all in the proof of Theorem 1. Thus, the proof of Theorem 1 can be adapted here.

For a 2-factor with more than two components, Theorems 1 and 2 can also be further extended after giving the following definitions.

Definition 5. Let $\mathcal{F}$ be a 2-factor of a cubic graph G.

(A) Let $\mathcal{F}'$ be a subset of components of $\mathcal{F}$ such that $\mathcal{F}' = \{C_1, C_2, ..., C_\tau\}$ contains precisely two odd components C_1 and C_2, and all others components C_i, for $i \neq 1, 2$, are of even order. The subset $\mathcal{F}'$ is $C_{(12+)}$-*linked* if there is a circuit $C_0 = e_1 \cdots e_r$ such that

(1) $V(C_0) \subseteq \bigcup_{C \in \mathcal{F}'} V(C)$;
(2) $E(C_0) \cap E(C_1) = \{e_{\ell_1}, e_{\ell_3}, e_{\ell_5}\}$, and, $E(C_0) \cap E(C_2) = \{e_{\ell_2}, e_{\ell_4}, e_{\ell_6}\}$ with

$$1 \leq \ell_1 < \ell_2 < \ell_3 < \ell_4 < \ell_5 < \ell_6 \leq r;$$

(3) $$|E(C_0) \cap E(C_i)| \leq 1$$

for every $i > 2$.

(B) The graph G is $\mathcal{F}$-$C_{(12+)}$-*linked* if the set of components of $\mathcal{F}$ has a partition $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_\omega\}$ such that *every member* $\mathcal{F}_i$ of the partition is $C_{(12+)}$-linked.

Theorems 1 and 2 can be further extended as follows.

Theorem 3. *Every $\mathcal{F}$-$C_{(12+)}$-linked cubic graph is Berge-Fulkerson colorable.*

The proof of Theorem 3 is omitted in this paper since it is almost the same as the proof of Theorems 1 and 2 with a slightly lengthy discussions when edges of M in C_0 are replaced with paths of odd length, each of which consists of some edges in M and some edges of even components.

ACKNOWLEDGMENTS

This study was supported by the National Science Foundation (No. DMS-1700218) and the National Natural Science Foundation of China (Nos. 11971054, 11731002).

ORCID

Siyan Liu http://orcid.org/0000-0001-6265-0429
Rong-Xia Hao http://orcid.org/0000-0001-8714-8750
Cun-Quan Zhang http://orcid.org/0000-0001-5583-4481

REFERENCES

1. M. Abreu, D. Labbate, R. Rizzi, and J. Sheehan, *Odd 2-factored snarks*, European J. Combin. **36** (2014), 460–472.
2. J. C. Bermond, B. Jackson, and F. Jaeger, *Shortest coverings of graphs with cycles*, J. Combin. Theory Ser. B **35** (1983), 297–308.
3. J. A. Bondy and U. S. R. Murty, Graph theory, Springer, New York, 2007.
4. G. Brinkmann, J. Goedgebeur, J. Hägglund, and K. Markström, *Generation and properties of snarks*, J. Combin. Theory Ser. B **103** (2013), 468–488.
5. U. A. Celmins, A study of three conjecture on an infinite family of snarks, Technical Report CORR-79-19, University of Waterloo, 1979.
6. F. Chen and G. Fan, *Fulkerson-covers of hypohamiltonian graphs*, Discrete Appl. Match. **186** (2015), 66–73.
7. G. Fan, *Covering graphs by cycles*, SIAM J. Discrete Math. **5** (1992), no. 4, 491–496.
8. G. Fan and A. Raspaud, *Fulkerson's conjecture and circuit covers*, J. Combin. Theory Ser. B **61** (1994), 133–138.
9. D. R. Fulkerson, *Blocking and anti-blocking pairs of polyhedra*, Math. Program. **1** (1971), 168–194.
10. J. Hägglund and A. Hoffmann-Ostenhof, *Construction of permutation snarks*, J. Combin. Theory Ser. B **122** (2017), 55–67.
11. R. Hao, J. Niu, X. Wang, C.-Q. Zhang, and T. Zhang, *A note on Berge-Fulkerson coloring*, Discrete Math. **309** (2009), 4235–4240.
12. R. Hao, C.-Q. Zhang, and T. Zheng, *Berge-Fulkerson coloring for $C_{(8)}$ -linked graphs*, J. Graph Theory **88** (2018), 46–60.
13. E. Máčajová and G. Mazzuoccolo, *Reduction of the Berge-Fulkerson conjecture to cyclically 5-edge-connected snarks*, Proc. Amer. Math. Soc. **148** (2020), 4643–4652.
14. P. Manuel and A. S. Shanthi, *Berge-Fulkerson conjecture on certain snarks*, Math. Comput. Sci. **9** (2015), 209–220.
15. G. Mazzuoccolo, *The equivalence of two conjectures of Berge and Fulkerson*, J. Graph Theory **68** (2011), 125–128.
16. P. D. Seymour, *On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte*, Proc. Lond. Math. Soc. **38** (1979), 423–460.
17. C. -Q. Zhang, Integer flows and cycle covers of graphs, Marcel Dekker Inc., New York, 1997.

How to cite this article: S. Liu, R. Hao, C. Zhang, and Z. Zhang, *Berge–Fulkerson coloring for $C_{(12)}$-linked permutation graphs*, J. Graph. Theory. (2021), 1–14.
https://doi.org/10.1002/jgt.22720