

SIGNED GRAPHS: FROM MODULO FLOWS TO INTEGER-VALUED FLOWS*

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Abstract. Converting modulo flows into integer-valued flows is one of the most critical steps in the study of integer flows. Tutte and Jaeger’s pioneering work shows the equivalence of modulo flows and integer-valued flows for ordinary graphs. However, such equivalence no longer holds for signed graphs. This motivates us to study how to convert modulo flows into integer-valued flows for signed graphs. In this paper, we generalize some early results by Xu and Zhang [*Discrete Math.*, 299 (2005), pp. 335–343], Schubert and Steffen [*European J. Combin.*, 48 (2015), pp. 34–47], and Zhu [*J. Combin. Theory Ser. B*, 112 (2015), pp. 93–103] and show that, for signed graphs, every modulo $(2 + \frac{1}{p})$ -flow with $p \in \mathbb{Z}^+ \cup \{\infty\}$ can be converted/extended into an integer-valued flow.

Key words. signed graph, integer flow, circular flow, modulo orientation

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1. Introduction. In flow theory, an integer-valued flow and a modulo flow are different by their definitions. For ordinary graphs, Tutte showed that *a graph admits an integer-valued nowhere-zero k -flow if and only if it admits a modulo nowhere-zero k -flow*. We also notice that although most landmark results are stated as integer-valued flow results, due to the theorem by Tutte, they were initially proved for modulo flows, such as the 8-flow theorem by Jaeger [5], the 6-flow theorem by Seymour [13], and the weak 3-flow theorem by Thomassen [15].

However, Tutte’s result cannot be applied for signed graphs (see Figure 1). That is, there is a big gap between modulo flows and integer-valued flows for signed graphs. The first known result was proved by Bouchet [1] in his study of chain-groups.

THEOREM 1.1 (see [1, Proposition 3.5]). *If a signed graph (G, σ) admits a modulo k -flow f_1 , then it admits an integer-valued $2k$ -flow f_2 with $\text{supp}(f_1) \subseteq \text{supp}(f_2)$.*

In this paper, Theorem 1.1 is improved for some important cases: modulo 2-flows, modulo 3-flows, and modulo circular $(2 + \frac{1}{p})$ -flows.

1.1. Basic definitions. Graphs considered here may have multiple edges or loops. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. For a vertex v , we denote by $E_G(v)$ the set of edges incident with v and denote $d_G(v) = |E_G(v)|$ (known as the *degree* of v). When no confusion is caused, we simply use $E(v)$ and $d(v)$ for short. Let X and Y be two disjoint vertex sets. We denote by $E(X, Y)$ the set of edges with one end in X and the other end in Y and denote $e(X, Y) = |E(X, Y)|$.

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An edge set F is an *odd- λ -edge cut* if $|F| = \lambda$ is odd and $G - F$ has more components than G . A graph G is *odd- λ -edge-connected* if it contains no odd- k -edge cut for any $k \leq \lambda - 2$. The *odd-edge-connectivity* of G is the smallest integer λ for which G is odd- λ -edge-connected. If $F = \{e\}$, then e is a *bridge* of G . A graph G is *bridgeless* if it contains no bridges.

A *signed graph* is a graph G associated with a *signature* $\sigma: E(G) \rightarrow \{\pm 1\}$. An edge e is *positive* if $\sigma(e) = 1$ and *negative* otherwise. Every edge of G consists of two half-edges, each of which is incident with exactly one end of this edge. For a vertex v , denote by $H(v)$ the set of all half-edges incident with v . Let $H(G) = \bigcup_{v \in V(G)} H(v)$. For a half-edge h , we use e_h to denote the edge containing h . An *orientation* of (G, σ) is a mapping $\tau: H(G) \rightarrow \{\pm 1\}$ such that $\tau(h_1)\tau(h_2) = -\sigma(e)$ for $e \in E(G)$, where h_1 and h_2 are the two half-edges of e .

For a signed graph (G, σ) , *switching* at a vertex u means reversing the signs of all edges incident with u . Let $\mathcal{X}_{(G, \sigma)}$ be the set of signatures of G obtained from σ via a sequence of switching operations. The *negativeness* of G is the smallest integer q for which G has a signature $\sigma' \in \mathcal{X}_{(G, \sigma)}$ with exactly q negative edges.

1.2. Integer-valued flows in signed graphs.

DEFINITION 1.2. Let (G, σ) be a signed graph associated with an orientation τ . Let k be a positive integer and $f: E(G) \rightarrow \mathbb{Z}$ be a mapping such that $0 \leq |f(e)| \leq (k - 1)$ for every edge $e \in E(G)$. The boundary of f at a vertex v is defined as $\partial f(v) = \sum_{h \in H(v)} f(e_h)\tau(h)$. The mapping f is an integer-valued k -flow (modulo k -flow, respectively) of (G, σ) if $\partial f(v) = 0$ ($\partial f(v) \equiv 0 \pmod{k}$, respectively) for each vertex $v \in V(G)$.

Let f be a flow of a signed graph (G, σ) . The *support* of f , denoted by $\text{supp}(f)$, is the set of edges e with $f(e) \neq 0$. A flow f is *nowhere-zero* if $\text{supp}(f) = E(G)$. For convenience, we respectively shorten the notations of nowhere-zero k -flows into integer-valued k -NZFs and modulo k -NZFs.

To verify Bouchet's 6-flow conjecture [1] for 6-edge-connected signed graphs, Xu and Zhang [18] proved the following two results, which generalize Tutte's theorem to a signed graph with $k = 2, 3$.

THEOREM 1.3 (see [18]). *If a signed graph (G, σ) admits a modulo 2-flow f_1 such that each component of $\text{supp}(f_1)$ contains an even number of negative edges, then it also admits an integer-valued 2-flow f_2 with $\text{supp}(f_1) = \text{supp}(f_2)$.*

THEOREM 1.4 (see [18]). *If a signed graph (G, σ) admits a modulo 3-flow f_1 such that $\text{supp}(f_1)$ is bridgeless, then it also admits an integer-valued 3-flow f_2 with $\text{supp}(f_1) = \text{supp}(f_2)$.*

In this paper, under the weaker conditions, we prove the following two results, which are analogues of Theorem 1.1 and respectively improve Theorems 1.3 and 1.4.

THEOREM 1.5. *If a signed graph (G, σ) is connected and admits a modulo 2-flow f_1 such that $\text{supp}(f_1)$ contains an even number of negative edges, then it also admits an integer-valued 3-flow f_2 with $\text{supp}(f_1) = \{e \in E(G): f_2(e) = \pm 1\}$.*

THEOREM 1.6. *If a signed graph (G, σ) is bridgeless and admits a modulo 3-flow f_1 , then it also admits an integer-valued 4-flow f_2 with $\text{supp}(f_1) \subseteq \{e \in E(G): f_2(e) = \pm 1, \pm 2\}$.*

By applying a lemma in [2] together with Theorems 1.5 and 1.6, one can show that every bridgeless signed graph admitting a nowhere-zero integer flow admits a

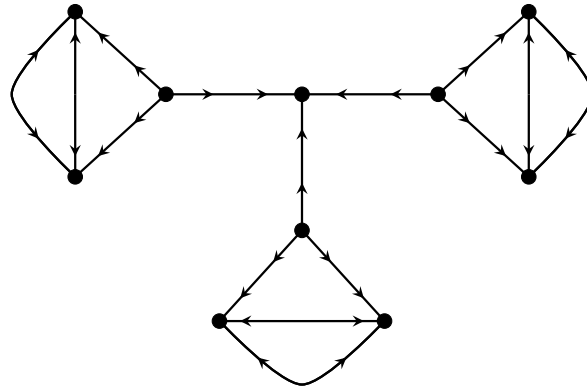


FIG. 1. (G, σ) admits a modulo 3-NZF with all edges assigned with 1 but no integer-valued 3-NZF.

nowhere-zero integer 11-flow.

1.3. Integer-valued circular flows in signed graphs.

DEFINITION 1.7. Let (G, σ) be a signed graph associated with an orientation τ .

- (1) Let k and d be two positive integers. An integer-valued (modulo, respectively) circular $\frac{k}{d}$ -flow of (G, σ) is an integer-valued (modulo, respectively) flow f such that $d \leq |f(e)| \leq k - d$ for every edge $e \in E(G)$.
- (2) Let p be a positive integer. The orientation τ is a modulo $(2p+1)$ -orientation if $\sum_{e \in H(v)} \tau(e) \equiv 0 \pmod{2p+1}$ for every vertex $v \in V(G)$.

When $k = 3$, Tutte's theorem [16] implies that a graph G admits a modulo circular 3-flow if and only if it admits an integer-valued circular 3-flow. This result was generalized to integer-valued circular $(2 + \frac{1}{p})$ -flows by Jaeger [6] as follows.

THEOREM 1.8 (see [6]). Let G be a graph. Then the following statements are equivalent:

- (A) G admits a modulo $(2p+1)$ -orientation.
- (B) G admits a modulo circular $(2 + \frac{1}{p})$ -flow.
- (C) G admits an integer-valued circular $(2 + \frac{1}{p})$ -flow.

For signed graphs, using an identical proof in [6], one can easily prove that (A) and (B) are still equivalent. However, similar to the argument for modulo flows, the equivalence relation between (B) and (C) does not hold for signed graphs (see Figure 1). For more details, readers are referred to [7], [8], [11], [12], [18], [20], etc.

The following are some early results proved by Xu and Zhang [18], Schubert and Steffen [12], and Zhu [20].

THEOREM 1.9. Let (G, σ) be a signed graph. Then (B) and (C) are equivalent if the following hold:

- (1) ([18]) $p = 1$, and (G, σ) is cubic and contains a perfect matching;
- (2) ([12]) (G, σ) is $(2p+1)$ -regular and contains a p -factor;
- (3) ([20]) (G, σ) is $(12p-1)$ -edge-connected with negativeness even or at least $(2p+1)$.

In this paper, we improve all the results in Theorem 1.9 as follows.

THEOREM 1.10. (B) and (C) are equivalent for signed graphs with odd-edge-connectivity at least $(2p+1)$. That is, if a signed graph (G, σ) is odd- $(2p+1)$ -connected,

then it admits a modulo circular $(2 + \frac{1}{p})$ -flow if and only if it admits an integer-valued circular $(2 + \frac{1}{p})$ -flow.

2. Proof of Theorem 1.5. Let (G, σ) together with a flow f_1 be a counterexample to Theorem 1.5 such that $|E(G)|$ is minimized. In the following context, we are to yield a contradiction by showing that (G, σ) actually admits an integer-valued 3-flow f_2 satisfying Theorem 1.5. For convenience, we denote $B = \text{supp}(f_1)$.

CLAIM 2.1. $B \neq E(G)$, and each edge of $E(G) - B$ is a bridge.

Proof. If $B = E(G)$, then G is an eulerian graph containing an even number of negative edges. By Theorem 1.3, G admits an integer-valued 2-NZF f_2 . If $e^* \in E(G) - B$ is not a bridge, let $G' = G - \{e^*\}$. Then G' is connected and f_1 is a modulo 2-flow of G' with $|E(G')| < |E(G)|$. Thus by the minimality of (G, σ) , (G', σ) admits an integer-valued 3-flow f_2 with $B = \{e \in E(G') : f_2(e) = \pm 1\}$. In both cases, f_2 is a desired integer-valued 3-flow. \square

CLAIM 2.2. For an edge $e \in E(G) - B$, denote the components of $G - \{e\}$ by Q_1 and Q_2 . Then each $B \cap Q_i$ contains an odd number of negative edges.

Proof. Since B contains an even number of negative edges, $B \cap Q_1$ and $B \cap Q_2$ contain the same parity number of negative edges. Suppose to the contrary that each contains an even number of negative edges. For $i \in \{1, 2\}$, we have $|E(Q_i)| < |E(G)|$, and therefore (Q_i, σ) admits an integer-valued 3-flow g_i such that $B \cap Q_i = \{e \in E(Q_i) : g_i(e) = \pm 1\}$. We define f_2 as $f_2(e') = g_i(e')$ for each $e' \in Q_i$ and $f_2(e) = 0$. It is easy to see that f_2 is a desired integer-valued 3-flow. \square

Now we first choose an edge e^* in $E(G) - B$ and denote its ends by x_1 and x_2 , respectively. For each $i \in \{1, 2\}$, let Q_i be the component of $G - \{e^*\}$ with $x_i \in V(Q_i)$. We construct a new signed graph (H_i, σ_i) from Q_i by adding a negative loop e_i at x_i . Denote $B_i = (B \cap Q_i) \cup \{e_i\}$, and assign $f_1(e_i) = 1$. By Claim 2.2, each B_i contains an even number of negative edges. Therefore, f_1 is a modulo 2-flow of (H_i, σ_i) with support B_i . Since $|E(H_i)| < |E(G)|$, by the minimality of G , (H_i, σ_i) admits an integer-valued 3-flow g_i such that $B_i = \{e \in E(H_i) : g_i(e) = \pm 1\}$. Note that $|\partial g_i(x_i)| = 2$ in Q_i . Without loss of generality, we can assume that $\partial g_2(x_2) = -\sigma(e^*)\partial g_1(x_1)$; otherwise we can replace g_1 by $-g_1$. Finally, we define f_2 by assigning $f_2(e) = g_i(e)$ for each $e \in E(Q_i)$, and by choosing $f_2(e^*) = 2$ or -2 such that the boundaries of f_2 at x_1 and x_2 are both zero. It is easy to verify that f_2 is a desired integer-valued 3-flow.

3. Proof of Theorem 1.6. First let us recall the vertex-splitting operation and the splitting lemma.

DEFINITION 3.1. Let G be a graph and v be a vertex. If $F \subset E_G(v)$, we denote by $G_{(v;F)}$ the graph obtained from G by splitting the edges of F away from v , that is, adding a new vertex v^* and changing the common end of edges in F from v to v^* (See Figure 2.)

LEMMA 3.2 (splitting lemma [3, 4]). Let G be a bridgeless graph and v be a vertex. If $d_G(v) \geq 4$ and $e_1, e_2, e_3 \in E_G(v)$ are chosen in a way that e_1 and e_3 are in different blocks when v is a cut-vertex, then either $G_{(v;\{e_1, e_2\})}$ or $G_{(v;\{e_1, e_3\})}$ is bridgeless. Furthermore, $G_{(v;\{e_1, e_3\})}$ is bridgeless if v is a cut-vertex.

Proof of Theorem 1.6. Let (G, σ) together with a flow f_1 be a counterexample to Theorem 1.6 such that the following hold:

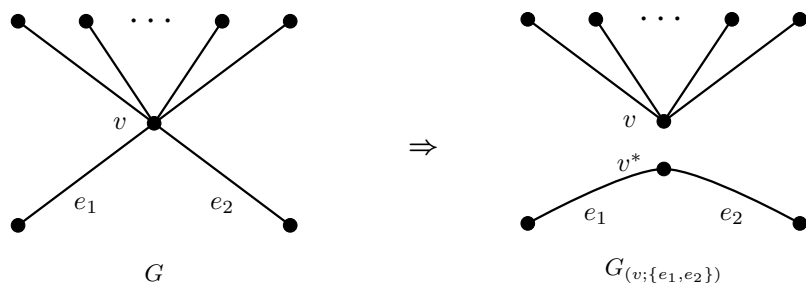


FIG. 2. Splitting $\{e_1, e_2\}$ away from v .

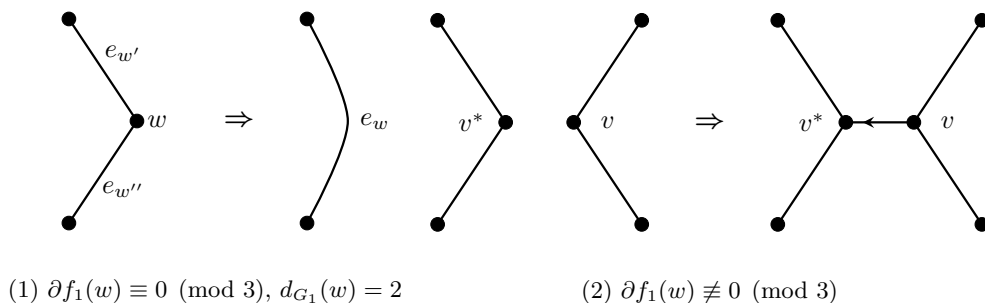


FIG. 3. Construction of signed graph (G_2, σ_2) .

- (1) $|\text{supp}^c(f_1)|$ is minimized, where $\text{supp}^c(f_1) = E(G) - \text{supp}(f_1)$;
- (2) subject to (1), $\sum_{v \in V(G)} |d_G(v) - 3|$ is minimized.

Now we use an argument similar to the one used in section 2 and show that (G, σ) actually admits an integer-valued 4-flow satisfying Theorem 1.6 in the following context.

CLAIM 3.1. $\text{supp}(f_1) \neq \emptyset$ and $\text{supp}^c(f_1) \neq \emptyset$.

Proof. If $\text{supp}(f_1) = \emptyset$, then simply let $f_2(e) = 0$ for each edge e . If $\text{supp}^c(f_1) = \emptyset$, then $\text{supp}(f_1) = E(G)$, and thus f_1 itself is a modulo 3-NZF of (G, σ) . Since G is bridgeless, Theorem 1.4 implies that (G, σ) admits an integer-valued 3-NZF f_2 . In both cases, f_2 is a desired integer-valued 4-flow. \square

CLAIM 3.2. The maximum degree of G is at most 3.

Proof. Suppose that G has a vertex v with $d_G(v) \geq 4$. Since G is bridgeless, Lemma 3.2 implies that we can split a pair of edges e_1, e_2 from v such that the resulting signed graph, say (G_1, σ_1) , is still bridgeless. In G_1 , we consider f_1 as a mapping on $E(G_1)$ and denote the common end of e_1 and e_2 by v^* . Thus, $\partial f_1(v^*) \equiv -\partial f_1(v) \pmod{3}$.

Let $w \in \{v, v^*\}$. If $\partial f_1(w) \equiv 0 \pmod{3}$ and $d_{G_1}(w) = 2$ with $E_{G_1}(w) = \{e_{w'}, e_{w''}\}$, then we further suppress the vertex w and denote the new edge by e_w (see Figure 3(1)). Then we can assign e_w with value $f_1(e_{w'})$, signature $\sigma_1(e_{w'})\sigma_1(e_{w''})$, and an orientation (based on its signature and value) in such a way that both ends of e_w have zero boundary. If $\partial f_1(w) \not\equiv 0 \pmod{3}$, then we further add a positive edge vv^* oriented from v to v^* and assign vv^* with value $\partial f_1(v^*)$ (see Figure 3(2)). In both cases, denote the resulting signed graph and mapping by (G_2, σ_2) and g_1 , respectively.

It is easy to verify that g_1 is a modulo 3-flow of (G_2, σ_2) and $|\text{supp}^c(g_1)| \leq |\text{supp}^c(f_1)|$ and that $\sum_{v \in V(G_2)} |d_{G_2}(v) - 3| < \sum_{v \in V(G)} |d_G(v) - 3|$. By the choice of (G, σ) , (G_2, σ_2) has an integer-valued 4-flow g_2 with $\text{supp}(g_1) \subseteq \{e \in E(G_2) : g_2(e) = \pm 1, \pm 2\}$. One can easily get a desired integer-valued 4-flow f_2 of (G, σ) from g_2 . \square

Note that G is connected. By Claim 3.1, G has a vertex x such that $E_G(x) \cap \text{supp}(f_1) \neq \emptyset$ and $E_G(x) \cap \text{supp}^c(f_1) \neq \emptyset$. Let e^* be an edge of $E_G(x) \cap \text{supp}^c(f_1)$, and denote the other end of e by y . We may without loss of generality assume that e^* is positive; otherwise we make a switch at x . We may further assume that e^* is oriented from x to y . Now we contract e^* and denote the resulting signed graph by (G', σ') . Thus, the restriction of f_1 to $E(G')$, say f'_1 , is a modulo 3-flow of (G', σ') . It follows from $\text{supp}(f'_1) = \text{supp}(f_1)$ that $|\text{supp}^c(f'_1)| < |\text{supp}^c(f_1)|$. Hence, (G', σ') admits an integer-valued 4-flow f'_2 such that $\text{supp}(f'_1) \subseteq \{e \in E(G') : f'_2(e) = \pm 1, \pm 2\}$.

Now we consider the mapping f'_2 on $E(G)$. Each vertex (possibly except x and y) has zero boundary and $\partial f'_2(x) = -\partial f'_2(y)$. If $\partial f'_2(x) \not\equiv 0 \pmod{3}$, then we extend f'_2 to a mapping h_1 by assigning $h_1(e^*) = -\partial f'_2(x)$. Thus, h_1 is a modulo 3-flow of G with $\text{supp}(h_1) \supset \text{supp}(f_1)$. This implies $|\text{supp}^c(h_1)| < |\text{supp}^c(f_1)|$, which contradicts the assumption (1). Thus, $\partial f'_2(x) \equiv 0 \pmod{3}$. In summary, x is a vertex satisfying $d_G(x) \leq 3$, $E_G(x) \cap \text{supp}^c(f_1) \neq \emptyset$, and $1 \leq |f'_2(e)| \leq 2$ for $e \in E_G(x) \cap \text{supp}(f_1)$. Hence, $0 \leq |\partial f'_2(x)| \leq 4$, and furthermore $|\partial f'_2(x)| \in \{0, 3\}$. Finally, we extend f'_2 to a mapping f_2 by assigning $f_2(e^*) = -\partial f'_2(x)$. Clearly, f_2 is an integer-valued 4-flow satisfying Theorem 1.6. \square

4. Proof of Theorem 1.10.

4.1. A new vertex splitting lemma. The vertex splitting method is one of the most useful techniques in graph theory (especially in the study of integer-valued flow and cycle cover problems). In section 3, we have discussed the splitting lemma introduced by Fleischner (see Lemma 3.2). Here are earlier results about vertex splitting by Nash-Williams [10], Mader [9], and Zhang [19].

THEOREM 4.1 (see [10]). *Let λ be an even integer and G be a λ -edge-connected graph. Let $v \in V(G)$, and let a be an integer such that $\lambda \leq a$ and $\lambda \leq d(v) - a$. Then there is an edge subset $F \subset E(v)$ such that $|F| = a$ and $G_{(v;F)}$ remains λ -edge-connected.*

THEOREM 4.2 (see [9]). *Let G be a graph, and let $v \in V(G)$ such that v is not a cut-vertex. If $d(v) \geq 4$ and v is adjacent to at least two distinct vertices, then there are two edges $e_1, e_2 \in E(v)$ such that, for every pair of vertices $x, y \in V(G) - \{v\}$, the local edge-connectivity between x and y in the graph $G_{(v; \{e_1, e_2\})}$ remains the same as in G .*

THEOREM 4.3 (see [19]). *Let G be a graph with odd-edge-connectivity at least λ_o . Let v be a vertex of G such that $d(v) \neq \lambda_o$ and $E(v) = \{e_0, e_1, \dots, e_{d(v)-1}\}$. Then there is a pair of edges $e_i, e_{i+1} \in E(v)$ (subindices modulo $d(v)$) such that the graph $G_{(v; \{e_i, e_{i+1}\})}$ remains odd- λ_o -edge-connected.*

DEFINITION 4.4. *Let G be a graph and v be a vertex. Let $S(v)$ be a subset of $\{(e_i, e_j) : e_i, e_j \in E(v) \text{ and } e_i \neq e_j\}$. The subset $S(v)$ is sequentially connected if, for every pair of edges $e', e'' \in E(v)$, there is a sequence $(e_0, e_1), (e_1, e_2), \dots, (e_{t-2}, e_{t-1}), (e_{t-1}, e_t) \in S(v)$ (subindices modulo $d(v)$) such that $e' = e_0$ and $e'' = e_t$.*

In Theorem 4.3, the subset $S(v) = \{(e_i, e_{i+1}) : i \in \mathbb{Z}_{d(v)}\}$ is sequentially connected. Therefore, the following theorem is a generalization of Theorem 4.3 and is

expected to have many applications in graph theory. The proof of Theorem 4.5 is identical to the one in [19], and an alternative proof can be also found in [14].

THEOREM 4.5. *Let G be a graph with odd-edge-connectivity at least λ_o and v be a vertex with $d(v) \neq \lambda_o$. Let $S(v)$ be a subset of $\{(e_i, e_j) : e_i, e_j \in E(v) \text{ and } e_i \neq e_j\}$. If the subset $S(v)$ is sequentially connected, then there is a pair of edges $(e', e'') \in S(v)$ such that the graph $G_{(v; \{e', e''\})}$ remains odd- λ_o -edge-connected.*

The following corollary is an analogue of Theorem 4.1 with respect to odd-edge-connectivity.

COROLLARY 4.6. *Let G be a graph with odd-edge-connectivity at least λ_o and v be a vertex with $d(v) > \lambda_o$. Let a be an even integer such that $a \leq d(v) - \lambda_o$. Then there is an edge subset $F \subset E(v)$ of size a such that $G_{(v; F)}$ remains odd- λ_o -edge-connected.*

Proof. Let $a = 2b$. Now we apply Theorem 4.5 to v repeatedly b times at v (taking $S(v)$ to be the set of all possible ordered pairs of $E(v)$ in each step). Then the resulting graph remains odd- λ_o -edge-connected. Denote by $\{v_1^*, \dots, v_b^*\}$ the set of the resulting vertices of degree two. It is easy to see that $\{vv_i^* : i = 1, \dots, b\}$ is a desired edge subset F of $E(v)$. \square

4.2. An application of Tutte's f -factor theorem. Theorem 1.10 will be proved by applying both Theorem 4.5 and some f -factor lemmas (such as Lemma 4.10) in this section.

DEFINITION 4.7. *Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}^+$ be a mapping. An f -factor of G is a subgraph H such that $d_H(v) = f(v)$ for each vertex $v \in V(G)$. In particular, if the range of f is $\{1, 2\}$, we simply call H a $\{1, 2\}$ -factor.*

In [17], Tutte gave a necessary and sufficient condition of the existence of f -factors.

THEOREM 4.8 (see [17]). *A graph G has an f -factor if and only if for any two disjoint vertex subsets $S, T \subseteq V(G)$,*

$$(4.1) \quad \sum_{v \in S} f(v) \geq |\mathcal{O}(S, T)| + \sum_{v \in T} [f(v) - d_{G-S}(v)],$$

where $\mathcal{O}(S, T)$ is the set of components U of $G - S - T$ for which

$$(4.2) \quad \sum_{v \in U} f(v) + e(U, T) \equiv 1 \pmod{2}.$$

Next we apply Tutte's f -factor theorem to find a $\{1, 2\}$ -factor for graphs defined below.

LEMMA 4.9. *Let k be an odd integer and G be an odd- k -edge-connected graph. Let $\{V_1, V_2\}$ be a partition of $V(G)$ such that $d_G(v) = k$ if $v \in V_1$ and $d_G(v) = 2k$ if $v \in V_2$. If f is a function satisfying $f(v) = d_G(v)/k$ for each vertex v , then G has an f -factor.*

Proof. Let S and T be two disjoint subsets of $V(G)$ and $\mathcal{O} = \mathcal{O}(S, T)$. Let $\{Q_1, Q_2, Q_3, Q_4\}$ be a partition of T , where for each $t \in \{1, 2\}$, Q_t consists of the vertices $v \in T \cap V_t$ such that $d_{G-S}(v) = 0$, Q_3 consists of the vertices v of $T \cap V_2$ such that $d_{G-S}(v) = 1$, and $Q_4 = T - Q_1 - Q_2 - Q_3$. The following claim directly follows from the definitions.

CLAIM 4.1.

- (1) $kf(v) = d_G(v)$ and $f(v) \equiv d_G(v) \pmod{2}$ for each vertex v .
- (2) $\sum_{v \in U} d_G(v) + e(U, T) \equiv 1 \pmod{2}$ for each $U \in \mathcal{O}$.

We partition \mathcal{O} into \mathcal{O}_1 and \mathcal{O}_2 , where

$$\mathcal{O}_1 = \{U \in \mathcal{O} : e(U, T) = 0\} \quad \text{and} \quad \mathcal{O}_2 = \{U \in \mathcal{O} : e(U, T) \neq 0\}.$$

CLAIM 4.2.

$$\sum_{U \in \mathcal{O}} e(U, S) \geq k|\mathcal{O}_1| + |\mathcal{O}_2|.$$

Proof. Note that if $U \in \mathcal{O}_1$, then $e(U, T) = 0$, and thus $E(U, S)$ is an edge-cut. Since G is odd- k -edge-connected, it suffices to show that for each $U \in \mathcal{O}$, $e(U, S) \equiv 1 \pmod{2}$.

For each $U \in \mathcal{O}$, we have

$$\sum_{v \in U} d_G(v) \equiv e(U, T) + e(U, S) \equiv -e(U, T) + e(U, S) \pmod{2}.$$

Thus by Claim 4.1(2), we have $e(U, S) \equiv 1 \pmod{2}$. □

CLAIM 4.3.

$$e(S, T) = \sum_{v \in T} [d_G(v) - d_{G-S}(v)] \geq k \sum_{v \in T} [f(v) - d_{G-S}(v)] + (k-1)|\mathcal{O}_2|.$$

Proof. Since $d_{G-S}(v) = 0$ if $v \in Q_1 \cup Q_2$ and $d_{G-S}(v) = 1$ if $v \in Q_3$, we have

$$(4.3) \quad \sum_{v \in Q_1 \cup Q_2 \cup Q_3} [d_G(v) - d_{G-S}(v)] = k \sum_{v \in Q_1 \cup Q_2 \cup Q_3} [f(v) - d_{G-S}(v)] + (k-1) \sum_{v \in Q_3} d_{G-S}(v).$$

Since $kf(v) = d_G(v)$ for each vertex v , we have

$$(4.4) \quad \sum_{v \in Q_4} [d_G(v) - d_{G-S}(v)] = \sum_{v \in Q_4} [kf(v) - d_{G-S}(v)] = k \sum_{v \in Q_4} [f(v) - d_{G-S}(v)] + (k-1) \sum_{v \in Q_4} d_{G-S}(v).$$

Combining (4.3) and (4.4), we have

$$(4.5) \quad \sum_{v \in T} [d_G(v) - d_{G-S}(v)] = k \sum_{v \in T} [f(v) - d_{G-S}(v)] + (k-1) \sum_{v \in Q_3 \cup Q_4} d_{G-S}(v).$$

Since each vertex $v \in Q_3 \cup Q_4$ is adjacent to at most $d_{G-S}(v)$ components in \mathcal{O}_2 , we have

$$(4.6) \quad \sum_{v \in Q_3 \cup Q_4} d_{G-S}(v) \geq |\mathcal{O}_2|.$$

Combining (4.5) and (4.6), we have

$$e(S, T) \geq k \sum_{v \in T} [f(v) - d_{G-S}(v)] + (k-1)|\mathcal{O}_2|. \quad \square$$

Denote $S^c = V(G) - S$. Now we are to estimate $e(S, S^c)$ in two ways by finding a lower bound and an upper bound. Obviously,

$$(4.7) \quad e(S, S^c) \leq \sum_{v \in S} d_G(v) = k \sum_{v \in S} f(v).$$

On the other hand,

$$(4.8) \quad e(S, S^c) \geq e(S, T) + \sum_{U \in \mathcal{O}} e(S, U).$$

By (4.7) and (4.8) together with Claims 4.2 and 4.3, we have

$$(4.9) \quad \begin{aligned} k \sum_{v \in S} f(v) &\geq k \sum_{v \in T} [f(v) - d_{G-S}(v)] + (k-1)|\mathcal{O}_2| + k|\mathcal{O}_1| + |\mathcal{O}_2| \\ &= k \sum_{v \in T} [f(v) - d_{G-S}(v)] + k(|\mathcal{O}_1| + |\mathcal{O}_2|) \\ &= k \left(\sum_{v \in T} [f(v) - d_{G-S}(v)] + |\mathcal{O}| \right). \end{aligned}$$

By (4.9), we have

$$\sum_{v \in S} f(v) \geq |\mathcal{O}| + \sum_{v \in T} [f(v) - d_{G-S}(v)].$$

Therefore, by Theorem 4.8, G has an f -factor. \square

LEMMA 4.10. *Let G be a graph with odd-edge-connectivity at least $(2p+1)$. If there is a mapping $\mu : V(G) \rightarrow \mathbb{Z}^+$ such that $d_G(v) = (2p+1)\mu(v)$ for each vertex $v \in V(G)$, then there is a spanning subgraph F such that $d_F(v) = p\mu(v)$.*

Proof. For each vertex v with $d_G(v) \notin \{2p+1, 2(2p+1)\}$, we first apply Corollary 4.6 to v with $a = 2(2p+1)$ and $\lambda_o = 2p+1$. Repeatedly apply this process until the degree of every vertex is either $(2p+1)$ or $2(2p+1)$. Let G' denote the resulting graph.

Next we apply Lemma 4.9 to G' with $k = 2p+1$. Let F_0 be a $\{1, 2\}$ -factor of G' such that, for each $v \in V(G')$, $d_{F_0}(v) = 1$ if $d_{G'}(v) = 2p+1$ and $d_{F_0}(v) = 2$ if $d_{G'}(v) = 2(2p+1)$.

Let $G'' = G' - E(F_0)$. Split each vertex v of G'' with $d_{G''}(v) = 4p$ into a pair of degree $2p$ vertices (no need to preserve the odd-edge-connectivity here). Let G''' be the resulting $2p$ -regular graph. By Petersen's theorem, G''' has a 2-factorization $\{F_1, \dots, F_p\}$.

When p is even, say $p = 2q$, the subgraph F induced by the edges of F_1, \dots, F_q is a desired spanning subgraph. When p is odd, say $p = 2q+1$, the subgraph F induced by the edges of F_0, F_1, \dots, F_q is a desired spanning subgraph. \square

4.3. Completion of the proof of Theorem 1.10. Now we are ready to complete the proof of Theorem 1.10.

It is obvious that (C) implies (B). Since (A) and (B) in Theorem 1.8 are equivalent, we will prove that (A) implies (C).

Let (G, σ) be an odd- $(2p+1)$ -edge-connected signed graph and τ be a modulo $(2p+1)$ -orientation of (G, σ) . We are going to show that (G, σ) has an integer-valued circular $(2 + \frac{1}{p})$ -flow.

For each $v \in V(G)$, denote $H_\tau^+(v) = \{h \in H(v) : \tau(v) = 1\}$ and $H_\tau^-(v) = \{h \in H(v) : \tau(v) = -1\}$. Let $d_\tau^+(v) = |H_\tau^+(v)|$ and $d_\tau^-(v) = |H_\tau^-(v)|$. If both $d_\tau^+(v) > 0$ and $d_\tau^-(v) > 0$ for some vertex v , then by Theorem 4.5 with $S(v) = \{(e', e'') : e' \in H_\tau^+(v) \text{ and } e'' \in H_\tau^-(v)\}$, one can split a pair of half-edges (one from $H_\tau^+(v)$ and the other from $H_\tau^-(v)$) away from v and then suppress the resulting degree 2 vertex.

Let G' be the resulting graph obtained from G by repeatedly applying Theorem 4.5 until no such pair of edges exists. Then G' remains odd- $(2p + 1)$ -edge-connected. Since τ remains a modulo $(2p + 1)$ -orientation of (G', σ) and either $d_{\tau}^{+}(v) = 0$ or $d_{\tau}^{-}(v) = 0$ for each vertex v of G' , there is a mapping μ of $G' : V(G') \rightarrow \mathbb{Z}^{+}$ such that $d_{G'}(v) = (2p + 1)\mu(v)$.

By Lemma 4.10, G' has a spanning subgraph F such that $d_F(v) = p\mu(v)$. Then the integer-valued function f^* defined as follows is a circular $(2 + \frac{1}{p})$ -flow of (G, σ) :

$$f^*(e) = \begin{cases} p & \text{if } e \notin F, \\ -p - 1 & \text{if } e \in F. \end{cases}$$

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