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FLOWS ON SIGNED GRAPHS WITHOUT LONG BARBELLS*

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Abstract. Many basic properties in Tutte's flow theory for unsigned graphs do not have their counterparts for signed graphs. However, signed graphs without long barbells in many ways behave like unsigned graphs from the point view of flows. In this paper, we study whether some basic properties in Tutte's flow theory remain valid for this family of signed graphs. Specifically let (G, σ) be a flow-admissible signed graph without long barbells. We show that it admits a nowhere-zero 6-flow and that it admits a nowhere-zero modulo k-flow if and only if it admits a nowhere-zero integer k-flow for each integer $k \geq 3$ and $k \neq 4$. We also show that each nowhere-zero positive integer k-flow of (G, σ) can be expressed as the sum of some 2-flows. For general graphs, we show that every nowhere-zero $\frac{p}{q}$-flow can be normalized in such a way, that each flow value is a multiple of $\frac{1}{2q}$. As a consequence we prove the equality of the integer flow number and the ceiling of the circular flow number for flow-admissible signed graphs without long barbells.

Key words. flows, long barbell, signed graphs, modulo flow, circular flow

AMS subject classifications. 05C22, 05C15

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1. Introduction.

Many basic properties in Tutte's flow theory for unsigned graphs do not have their counterparts for signed graphs. For instance, Tutte's 5-flow conjecture [24] states that every flow-admissible unsigned graph has a nowhere-zero 5-flow. The best approximation so far is that every flow-admissible unsigned graph has a nowhere-zero 6-flow [18]. Flow-admissible signed graphs which do not admit a nowhere-zero 5-flow are known. Therefore, the 5-flow conjecture is not true for signed graphs in general. But a 6-flow theorem might be true for flow-admissible signed graphs as conjectured by Bouchet [1]. This conjecture is verified for several classes of signed graphs (see, e.g., [5, 6, 9, 13, 16, 17, 25]).

The signed graphs without long barbells form a very interesting family in general. Slilaty [20] presents a complete characterization of signed graphs without long barbells (Theorem 1.2 in [20]). Such a signed graph can also be translated into a special unsigned graph without vertex-disjoint odd circuits by inserting one vertex of degree 2 into each positive edge. Readers are referred to [7] and [19] for a characterization of unsigned graphs without vertex-disjoint odd circuits.

The family of signed graphs without long barbells also has its special interest from the point view of flow theory. It is well known that cycles are fundamental elements in flow theory. For unsigned graphs, every element in the cycle space is the support of

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a 2-flow. However, some element (long barbells) in the cycle space of a signed graph is the support of a 3-flow but not a 2-flow. Therefore, we may expect signed graphs without long barbells to inherit some nice properties from unsigned graphs, which naturally motivates the question whether signed graphs without long barbells have almost similar properties as unsigned graphs in Tutte's flow theory. Unfortunately, the answer is no. For example, the unsigned Petersen graph admits a nowhere-zero 5-flow, while the signed Petersen graph of Figure 1, which has no long barbells, admits a nowhere-zero 6-flow but no nowhere-zero 5-flow.

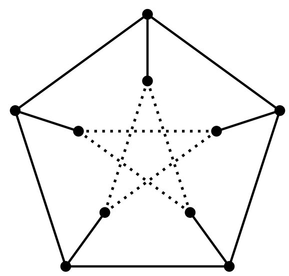

Fig. 1. *A signed Petersen graph admits a nowhere-zero* 6*-flow but no nowhere-zero* 5*-flow. Positive edges are solid, and negative edges are dashed.*

Khelladi verified Bouchet's 6-flow conjecture for flow-admissible 3-edge-connected signed graphs without long barbells.

Theorem 1.1 (Khelladi [6]). *Let (G, σ) be a flow-admissible 3-edge-connected signed graph. If (G, σ) contains no long barbells, then it admits a nowhere-zero 6-flow.*

Lu et al. [9] also showed that every flow-admissible cubic signed graph without long barbells admits a nowhere-zero 6-flow. In section 3 we will verify Bouchet's 6-flow conjecture for the family of signed graphs without long barbells. We further study the relation between modulo flows and integer flows on signed graphs. The equivalency of modulo flow and integer flow is a fundamental result in the theory of flows on unsigned graphs.

Theorem 1.2 (Tutte [23], or see Younger [27]). *An unsigned graph admits a nowhere-zero modulo k-flow if and only if it admits a nowhere-zero k-flow.*

Almost all landmark results in flow theory, such as the 4-flow and 8-flow theorems by Jaeger [4], the 6-flow theorem by Seymour [18], the 3-flow theorems by Thomassen [22] and by Lovász et al. [11], are proved for modulo flows.

However, there is no equivalent result in regard to Theorem 1.2 for signed graphs in general.

We will prove an analog of Theorem 1.2 for the family of signed graphs without long barbells. We show that the admittance of a nowhere-zero modulo k-flow and a nowhere-zero k-flow are equivalent for $k = 3$ or $k \geq 5$.

In section 4 we study the decomposition of flows. For unsigned graphs, a positive k-flow can be expressed as the sum of some 2-flows.

Theorem 1.3 (Little, Tutte, and Younger [8]). *Let G be an unsigned graph and (τ, f) be a positive k-flow of G. Then*

$$(\tau, f) \;=\; \sum_{i=1}^{k-1} (\tau, f_i),$$

where each (τ, f_i) is a nonnegative 2-flow.

We extend Theorem 1.3 to the class of signed graphs without long barbells.

The paper closes with the study of circular flows in section 5. For an unsigned graph G, Goddyn et al. [2] showed $\Phi_i(G) = \lceil \Phi_c(G) \rceil$. Raspaud and Zhu [15] conjectured this to be true for a signed graph (G, σ) as well, and they proved that $\Phi_i(G, \sigma) \leq 2\lceil \Phi_c(G, \sigma) \rceil - 1$. The conjecture was disproved in [17] by constructing a family of signed graphs where the supremum of $\Phi_i(G, \sigma) - \Phi_c(G, \sigma)$ is 2 (see one member of the family depicted in Figure 5). This result was further improved in [14] by showing that the supremum of $\Phi_i(G, \sigma) - \Phi_c(G, \sigma)$ is 3 which is best possible if Bouchet's 6-flow conjecture is true. We show that $\Phi_i(G, \sigma) = \lceil \Phi_c(G, \sigma) \rceil$ for a signed graph (G, σ) without long barbells and verify the conjecture of Raspaud and Zhu for this family of signed graphs. The result is a consequence of a normalization theorem for signed graphs which states that every nowhere-zero $\frac{p}{q}$-flow on a signed graph can be normalized in such a way, that each flow value is a multiple of $\frac{1}{2q}$. For unsigned graphs it is known [21] that every nowhere-zero $\frac{p}{q}$-flow on a signed graph can be normalized in such a way that each flow value is a multiple of $\frac{1}{q}$. We show that this is also true for signed graphs without long barbells.

2. Notations and terminology.

Let G be a graph. For $S \subseteq V(G)$, the set $V(G) - S$ is denoted by S^c. For $U_1, U_2 \subseteq V(G)$, the set of edges with one end in U_1 and the other in U_2 is denoted by $\delta_G(U_1, U_2)$. For convenience, we write $\delta_G(U_1)$ for $\delta_G(U_1, U_1^c)$ and $\delta_G(v)$ for $\delta_G(\{v\})$. The degree $d_G(v)$ of v is the number of edges incident with v where a loop is counted twice.

A *signed graph* (G, σ) is a graph G together with a *signature* $\sigma : E(G) \to \{-1, 1\}$. An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* otherwise. The set $E_N(G, \sigma)$ denotes the set of all negative edges in (G, σ). An unsigned graph can also be considered as a signed graph with the all-positive signature; i.e., $E_N(G, \sigma) = \emptyset$. Let (G, σ) be a signed graph. A path P in G is called a *subdivided edge* of G if every internal vertex of P is a 2-vertex. The *suppressed graph* of G, denoted by $\overline{G}$, is the signed graph obtained from G by replacing each maximal subdivided edge P with a single edge e and assigning $\sigma(e) = \sigma(P)$ where $\sigma(P)$ is the product of the signs of the edges in $E(P)$. A circuit $(C, \sigma|_{E(C)})$, or shortly C, is a connected 2-regular subgraph of (G, σ). A circuit C is *balanced* if $|E_N(C)| \equiv 0 \pmod 2$, and it is *unbalanced* otherwise. A signed graph is *balanced* if it does not contain an unbalanced circuit, and it is *unbalanced* otherwise. A *signed circuit* is a signed graph of one of the following three types:

(1) a balanced circuit;
(2) a short barbell, the union of two unbalanced circuits that meet at a single vertex;
(3) a long barbell, the union of two disjoint unbalanced circuits with a path that meets the circuits only at its ends.

Following Bouchet [1], we view an edge $e = uv$ of a signed graph (G, σ) as two *half-edges* h_e^u and h_e^v, one incident with u and one incident with v. Let $H_G(v)$ (abbreviated $H(v)$) be the set of all half-edges incident with v, and $H(G)$ be the set of all half-edges in (G, σ). An *orientation* of (G, σ) is a mapping $\tau : H(G) \to \{-1, +1\}$ such that for every $e = uv \in E(G)$, $\tau(h_e^u)\tau(h_e^v) = -\sigma(e)$. If $\tau(h_e^u) = 1$, then h_e^u is oriented away from u; if $\tau(h_e^u) = -1$, then h_e^u is oriented toward u. Thus, based on the signature, a positive edge can be directed like $\bullet\!\rightarrow\!\!-\!\!\rightarrow\!\bullet$ or like $\bullet\!\leftarrow\!\!-\!\!\leftarrow\!\bullet$, and a negative edge can be directed like $\bullet\dashrightarrow\dashleftarrow\bullet$ or like $\bullet\dashleftarrow\dashrightarrow\bullet$. A signed graph (G, σ) together with an orientation τ is called an *oriented signed graph*, denoted by (G, τ), with underlying signature σ_τ.

Definition 2.1. *Let (G, τ) be an oriented signed graph and $f : E(G) \to \mathbb{R}$ be a mapping. Let $r \geq 2$ be a real number and $k \geq 2$ be an integer.*

(1) *The* boundary *of (τ, f) is the mapping $\partial(\tau, f) : V(G) \to \mathbb{R}$ defined as*

$$\partial(\tau, f)(v) = \sum_{h \in H(v)} \tau(h) f(e_h)$$

for each vertex v, where e_h is the edge of (G, σ_τ) containing h.

(2) *The* support *of f, denoted by* $\text{supp}(f)$*, is the set of edges e with $|f(e)| > 0$.*

(3) *If $\partial(\tau, f) = 0$, then (τ, f) is called a* flow *of (G, σ_τ). A flow (τ, f) is said to be* nowhere-zero *of (G, σ_τ) if $\text{supp}(f) = E(G)$.*

(4) *If $1 \leq |f(e)| \leq r - 1$ for each $e \in E(G)$, then the flow (τ, f) is called a* circular r-flow *of (G, σ_τ).*

(5) *If $f(e) \in \mathbb{Z}$ and $1 \leq |f(e)| \leq k - 1$ for each $e \in E(G)$, then the flow (τ, f) is called a* nowhere-zero k-flow *of (G, σ_τ).*

(6) *If $\partial(\tau, f) \equiv 0 \pmod{k}$ and $f(e) \in \mathbb{Z}_k \setminus \{0\}$ for each $e \in E(G)$, then the flow (τ, f) is called a* nowhere-zero modulo k-flow *or a* nowhere-zero $\mathbb{Z}_k$-flow *of (G, σ_τ).*

A signed graph is *flow-admissible* if it admits a nowhere-zero k-flow for some integer k. In a signed graph, *switching* at a vertex u means reversing the signs of all edges incident with u. Two signed graphs are *equivalent* if one can be obtained from the other by a sequence of switches. Then a signed graph is balanced if and only if it is equivalent to a graph without negative edges. Note that switching at a vertex does not change the parity of the number of negative edges in a circuit, and although technically it changes the flows, it only reverses the directions of the half-edges incident with the vertex and the directions of other half-edges and the flow values of all edges remain the same. Bouchet [1] gave a characterization for flow-admissible signed graphs.

Proposition 2.2 (Bouchet [1]). *A connected signed graph (G, σ) is flow-admissible if and only if it is not equivalent to a signed graph with exactly one negative edge and it has no cut-edge b such that $(G - b, \sigma|_{G-b})$ has a balanced component.*

The following lemma is a direct consequence of Proposition 2.2 and the definition of long barbell.

Lemma 2.3. *Let (G, σ) be a signed graph without long barbells. Then for each $X \subseteq V(G)$, one of $(G[X], \sigma|_{E(G[X])})$ and $(G[X^c], \sigma|_{E(G[X^c])})$ is balanced. Thus, if (G, σ) is flow-admissible, then (G, σ) is bridgeless.*

For a flow-admissible signed graph (G, σ), its *circular flow number* and *integer flow number* are defined, respectively, by

$$\Phi_c(G, \sigma) = \inf\{r : (G, \sigma) \text{ admits a circular } r\text{-flow}\},$$
$$\Phi_i(G, \sigma) = \min\{k : (G, \sigma) \text{ admits a nowhere-zero } k\text{-flow}\}.$$

Raspaud and Zhu [15] showed that $\Phi_c(G, \sigma)$ is a rational number for any flow-admissible signed graph (G, σ) and $\Phi_c(G, \sigma) = \min\{r : (G, \sigma) \text{ admits a circular } r\text{-flow}\}$, just like for unsigned graphs.

3. Integer flows and modulo flows.

3.1. Integer flows.

This subsection will extend Khelladi's result (Theorem 1.1) to the class of all flow-admissible signed graphs without long barbells. For the proof of our result we will need the following two results.

THEOREM 3.1 (Seymour [18]). *Every bridgeless unsigned graph admits a nowhere-zero 6-flow.*

LEMMA 3.2 (Lu, Luo, and Zhang [9]). *Let G be an unsigned graph with an orientation τ and assume that G admits a nowhere-zero k-flow. If a vertex u of G has degree at most 3 and $\gamma : \delta_G(u) \to \{\pm 1, \ldots, \pm(k-1)\}$ satisfies $\partial(\tau, \gamma)(u) = 0$, then there is a nowhere-zero k-flow (τ, ϕ) of G so that $\phi|_{\delta(u)} = \gamma$.*

THEOREM 3.3. *Let (G, σ) be a flow-admissible signed graph. If (G, σ) contains no long barbells, then it admits a nowhere-zero 6-flow.*

Proof. Suppose to the contrary that the statement is not true. Let (G, σ) be a counterexample with $|E(G)|$ minimum. We will deduce a contradiction to Theorem 1.1 by showing that G is 3-edge-connected.

We first show that the minimum degree of G, $\delta(G) \geq 3$. If G has vertices of degree two, then the suppressed graph $\overline{G}$ remains flow-admissible and contains no long barbells. Thus by the minimality of G, $\overline{G}$ admits a nowhere-zero 6-flow, so does G, a contradiction. Hence G contains no vertices of degree two. Since (G, σ) is flow-admissible, it contains no vertices of degree one and thus the minimum degree of G is at least three.

Next we show that G is 3-edge-connected. By Lemma 2.3, (G, σ) is bridgeless since it contains no long barbells.

Suppose that (G, σ) has a 2-edge-cut, say $\{u_1u_2, w_1w_2\}$. Since the minimum degree of G is at least 3, every 2-edge-cut is nontrivial. Let $(G_1, \sigma|_{E(G_1)})$ and $(G_2, \sigma|_{E(G_2)})$ be the two components of $G - \{e_1, e_2\}$, where $e_1 = u_1u_2$ and $e_2 = w_1w_2$ with $u_i, w_i \in V(G_i)$ for $i = 1, 2$. By Lemma 2.3 again, one of $(G_1, \sigma|_{E(G_1)})$ and $(G_2, \sigma|_{E(G_2)})$ is balanced. Without loss of generality we assume that $(G_1, \sigma|_{E(G_1)})$ is balanced. By switching, we may further assume that all edges in $(G_1, \sigma|_{E(G_1)})$ are positive. Fix an arbitrary τ on $H(G)$. Let G'_1 be the unsigned graph obtained from (G, σ) by contracting $H(G_2) \cup \{h_{e_1}^{u_2}, h_{e_2}^{w_2}\}$ into a vertex v_1, and let $(G'_2, \sigma|_{E(G'_2)})$ be the signed graph obtained from (G, σ) by contracting $H(G_1)$ into a vertex v_2. An illustration on G'_1 and $(G'_2, \sigma|_{E(G'_2)})$ is shown in Figure 2.

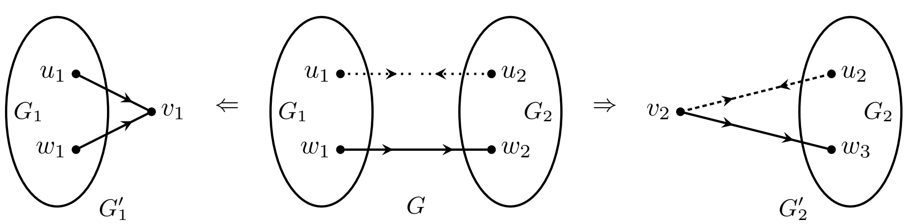

FIG. 2. *An illustration on how to construct G'_1 and $(G'_2, \sigma|_{E(G'_2)})$ from (G, σ).*

From the definition of $(G'_2, \sigma|_{E(G'_2)})$, we know that $(G'_2, \sigma|_{E(G'_2)})$ is flow-admissible and contains no long barbells. So $(G'_2, \sigma|_{E(G'_2)})$ admits a nowhere-zero 6-flow $(\tau|_{H(G'_2)}, f_2)$ by the minimality of (G, σ). Assign $\gamma(v_1u_1) = f_2(v_2u_2)$ and $\gamma(v_1w_1) = f_2(v_2w_2)$. Since G'_1 is an unsigned graph, the restriction of τ on $H(G_1) \cup \{h_{e_1}^{u_1}, h_{e_2}^{w_1}\}$ can be considered as an orientation of G'_1 denoted by τ_1. Then we have $\partial(\tau_1, \gamma)(v_1) = \partial(\tau|_{H(G'_2)}, f_2)(v_2) = 0$. By Theorem 3.1 and Lemma 3.2, there is a nowhere-zero 6-flow (τ_1, f_1) of G'_1 such that $f_1|_{\delta_{G'_1}(v_1)} = \gamma = f_2|_{\delta_{G'_2}(v_2)}$. Thus (τ_1, f_1) and $(\tau|_{H(G'_2)}, f_2)$

can be combined to a nowhere-zero 6-flow of (G, σ), a contradiction. Therefore G is 3-edge-connected, a contradiction to Theorem 1.1 since (G, σ) is a counterexample. ∎

3.2. From modulo flows to integer flows. In flow theory, an integer flow and a modulo flow are different by their definitions, but they are equivalent for unsigned graphs as shown by Tutte [24] (see Theorem 1.2). However, Tutte's result cannot be extended to signed graphs (see, e.g., [26]). That is, there is a gap between modulo flows and integer flows for signed graphs.

In this subsection, we will extend Tutte's result and show that the equivalence between nowhere-zero $\mathbb{Z}_k$-flows and nowhere-zero k-flows still holds for signed graphs without long barbells when $k = 3$ or $k \geq 5$.

THEOREM 3.4. *Let (G, σ) be a signed graph without long barbells, and let k be an integer with $k = 3$ or $k \geq 5$. Then (G, σ) admits a nowhere-zero $\mathbb{Z}_k$-flow if and only if it admits a nowhere-zero k-flow.*

The "if" part of Theorem 3.4 is trivial since every nowhere-zero k-flow is also a nowhere-zero $\mathbb{Z}_k$-flow in a signed graph. For the "only if" part of Theorem 3.4, by Lemma 2.3, the case of $k = 3$ is an immediate corollary of a result about $\mathbb{Z}_3$-flow in [26], and the case of $k \geq 6$ follows from Theorem 3.3, and thus we only need to consider the case of $k = 5$, which is a corollary of the following stronger result.

THEOREM 3.5. *Let $k \geq 3$ be an odd integer and (G, σ) be a signed graph with a nowhere-zero $\mathbb{Z}_k$-flow (τ, f_1). If (G, σ) does not contain a long barbell, then there is a nowhere-zero k-flow (τ, f_2) such that $f_1(e) \equiv f_2(e) \pmod{k}$.*

In order to prove Theorem 3.5, we introduce some new concepts.

DEFINITION 3.6. *Let $W = x_0e_1x_1e_2x_2 \dots e_{t-1}x_{t-1}e_tx_t$ be a signed walk with an orientation τ.*

(1) *W is called a* diwalk *from x_0 to x_t if $\tau(h_{e_1}^{x_0}) = 1$ and $\tau(h_{e_i}^{v_i}) + \tau(h_{e_{i+1}}^{v_i}) = 0$ for each $i \in \{1, \dots, t-1\}$.*
(2) *The diwalk W from x_0 to x_t is* positive *if $\tau(h_{e_t}^{x_t}) = -1$. Otherwise, it is* negative.
(3) *A diwalk is* all-positive *if all its edges are positive.*
(4) *A* ditrail *from x to y is a diwalk from x to y without repeated edges.*
(5) *A* dipath *from x to y is a diwalk from x to y without repeated vertices (see Figure 3).*

FIG. 3. (a) *A positive dipath from x_1 to x_5;* (b) *A negative dipath from x_1 to x_5.*

DEFINITION 3.7. *An oriented signed graph is called a* tadpole *with tail end x (see Figure 4) if*

(1) *it consists of a ditrail C and a dipath P with $V(C) \cap V(P) = \{v_1\}$;*
(2) *P is a positive dipath from x to v_1;*
(3) *C is a closed negative ditrail from v_1 to v_1.*

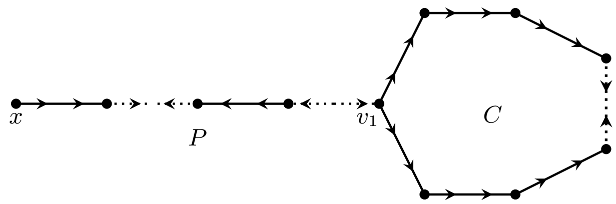

FIG. 4. *A tadpole with tail end x.*

Note that it is possible that $x = v_1$ in the above definition. In this case, the tadpole is called a *tailless tadpole*. Although in the proof of Theorem 3.5, the ditrail C of the tadpole is a ditrail without repeated vertices, the definition of a tadpole only requires C to be a ditrail which allows repeated vertices for general purpose.

DEFINITION 3.8. *Let (G, τ) be an oriented signed graph and $f : E(G) \to \mathbb{R}$.*

(1) *A vertex x is a* source *(resp.,* sink*) of (τ, f) if $\partial(\tau, f)(x) > 0$ (resp., $\partial(\tau, f)(x) < 0$).*

(2) *An edge e is a* source *(resp.,* sink*) of (τ, f) if the boundary at e, $\partial(\tau, f)(e) = -(\tau(h_1) + \tau(h_2))f(e)$, is positive (resp., negative), where h_1 and h_2 are the two half-edges of e.*

Note that an edge is a source or a sink if and only if it is negative. A sink is either a sink vertex or a sink edge and a source is either a source vertex or a source edge.

The following observation is a trivial fact in network theory.

OBSERVATION 3.9. *Let (G, τ) be an oriented signed graph and $f : E(G) \to \mathbb{R}$. The total sum of boundaries on $V(G) \cup E(G)$ is zero. In particular, if f is a flow, then the total sum of the boundaries on $E(G)$ is zero.*

The following observation is also a trivial fact in network theory which will be applied to find a tadpole.

OBSERVATION 3.10. *Let (G, τ) be an oriented signed graph and $f : E(G) \to \mathbb{R}^+ \cup \{0\}$. For each source x, there must exist a sink t_x such that there is an all-positive dipath from x to t_x.*

DEFINITION 3.11. *Let (G, τ) be an oriented signed graph, $E_0 \subseteq E(G)$, and $f : E(G) \to \mathbb{Z}_k$ be a mapping. The operation* minusing *of (τ, f) on E_0 is done by reversing the directions of both half-edges of e and changing $f(e)$ to $k - f(e)$ for every $e \in E_0$. The resulting pair obtained from (τ, f) is denoted by $(\tau_{\widetilde{E}_0}, f_{\widetilde{E}_0})$.*

We are ready to prove Theorem 3.5.

Proof of Theorem 3.5. Let (G, σ_0) be a counterexample and (τ_0, f_1) be a nowhere-zero $\mathbb{Z}_k$-flow of (G, σ_0). We can choose a triple (G, τ, f) obtained from (G, τ_0, f_1) by a sequence of switching and minusing operations such that

(S1) $0 < f(e) < k$ for every $e \in E(G)$;

(S2) subject to (S1), $\partial(\tau, f)(v) \equiv 0 \pmod k$ for every $v \in V(G)$;

(S3) subject to (S1) and (S2), $\eta(\tau, f) = \sum_{v \in V(G)} |\partial(\tau, f)(v)|$ is as small as possible;

(S4) subject to (S1), (S2), and (S3), the number of source vertices of (τ, f) is as large as possible.

Let $X = \{x \in V(G) : \partial(\tau, f)(x) > 0)\}$ be the set of source vertices of (τ, f). The following claim shows that by the choice of (G, τ, f), there is no sink vertex in (τ, f).

CLAIM 1. $X = \{x \in V(G) : \partial(\tau, f)(x) \neq 0)\}$. *That is, there is no sink vertex in (τ, f).*

Proof. Suppose to the contrary that there is a vertex $v \in V(G)$ such that $\partial(\tau, f)(v) < 0$. Let (G, τ') be the resulting oriented signed graph obtained from (G, τ) by switching at v, and let $X' = X \cup \{v\}$. Note that switching at v is done by reversing all directions of half-edges in $H_G(v)$. Thus (G, τ', f) satisfies (S1)–(S3) and X' is the set of source vertices of (τ', f). This contradicts (S4). ∎

The following claim shows that $\eta(\tau, f) \neq 0$, and thus (G, τ, f) is indeed a network with sinks and sources.

Claim 2. $X \neq \emptyset$.

Proof. Suppose $X = \emptyset$. Then (τ, f) is a nowhere-zero k-flow of the signed graph (G, σ). Since (G, τ, f) is obtained from (G, τ_0, f_1) by a sequence of switching and minusing operations, there are $V_0 \subseteq V(G)$, $E_0 \subseteq E(G)$ and an orientation τ_1 of (G, σ) such that (G, τ_1) is obtained from (G, τ_0) by switching on V_0, and (τ, f) is obtained from (τ_1, f_1) by minusing on E_0. Let $f' : E(G) \to \mathbb{Z}$ be defined as follows:

$$f'(e) = \begin{cases} f(e) & \text{if } e \notin E_0; \\ -f(e) & \text{if } e \in E_0. \end{cases}$$

Since (τ, f) is a nowhere-zero k-flow of (G, σ) and is obtained from (τ_1, f_1) by minusing on E_0, (τ_1, f') is also a nowhere-zero k-flow of (G, σ) and satisfies $f'(e) \equiv f_1(e) \pmod{k}$ for every $e \in E(G)$. Thus (τ_0, f') is a desired nowhere-zero k-flow of (G, σ_0) since (G, τ_1) is obtained from (G, τ_0) by switching on V_0. This contradicts that $(G, \sigma|_0)$ is a counterexample. ∎

By (S2) and Claim 1, every vertex x in X satisfies

$$\partial(\tau, f)(x) = \mu k$$

for some positive integer μ.

For directed unsigned graph, there is only one type of ditrails/dipaths. However, for directed signed graphs, there are two types of ditrails/dipaths, namely, positive and negative. We first show that a negative ditrail between two vertices in X does not exist in (G, τ).

Claim 3. *There is no negative ditrail of (G, τ) between two distinct vertices in X.*

Proof. Suppose to the contrary that X contains two distinct vertices x_1 and x_2 such that there exists a negative ditrail P from x_1 to x_2 in (G, τ). By the definition of negative ditrails (see Definition 3.6) and by Definition 3.11, it is not difficult to check that

$$\eta(\tau_{\widetilde{E(P)}}, f_{\widetilde{E(P)}}) = \sum_{i=1}^{2} (\partial(\tau, f)(x_i) - k) + \sum_{v \in V(G) \setminus \{x_1, x_2\}} \partial(\tau, f)(v) = \eta(\tau, f) - 2k.$$

This contradicts (S3). ∎

Similar to unsigned graphs, for a given source vertex $x \in X$ we need to study the properties of the graph induced by the vertices y in (G, τ) such that there is a dipath from x to y. We may partition such reachable vertices according to the signs of the dipath.

Pick an arbitrary vertex x from X by Claim 2, and let

$$
\begin{aligned}
&Y_x^+ = \{y \in V(G) : (G,\tau) \text{ contains a positive dipath from } x \text{ to } y\},\\
&Y_x^- = \{y \in V(G) : (G,\tau) \text{ contains a negative dipath from } x \text{ to } y\} \setminus Y_x^+, \text{ and}\\
&Y_x = Y_x^+ \cup Y_x^-.
\end{aligned}
$$

In fact, we will show that we may further assume that $Y_x^- = \emptyset$. By Claim 3, $Y_x^- \cap X = \emptyset$, so $\partial(\tau, f)(y) = 0$ for each $y \in Y_x^-$. Switch at every vertex in Y_x^-, and denote the resulting pair obtained from (G,τ) by (G,τ_1). Then (G,σ_{τ_1}) is equivalent to (G,σ_τ), and τ_1 is an orientation of (G,σ_{τ_1}). Since $\partial(\tau,f)(y) = 0$ for $y \in Y_x^-$, it is easy to see that the triple (G,τ_1,f) also satisfies (S1)–(S4). Moreover, by the definitions of Y_x^+ and Y_x^-, (G,τ_1) contains a positive dipath from x to y for every $y \in Y_x$. Without loss of generality, we can assume

$$
Y_x^- = \emptyset \quad \text{and} \quad Y_x = Y_x^+ \tag{1}
$$

and consider $(G,\tau_1,f) = (G,\tau,f)$. Then the following claim holds which will be applied to find tadpoles in $(G[Y_x],\tau)$.

Claim 4. *For every $y \in Y_x$, (G,τ) contains a positive dipath from x to y.*

Claim 5. *$(G[Y_x],\tau)$ contains a tadpole with tail end x (see Definition 3.7).*

Proof. By Observation 3.10, there is a sink t_x of (τ,f) such that (G,τ) contains an all-positive dipath from x to t_x. Note that (τ,f) contains no sink vertices by Claim 1. Hence t_x must be a sink edge, say $t_x = u'u''$. Let P_x' be an all-positive dipath from x to u'. Then $u' \in Y_x$, $t_x \notin E(P_x')$, and $P_x' + t_x$ is a negative dipath from x to u'' since t_x is a sink edge. Thus $u'' \in Y_x = Y_x^+$ (by (1)).

This implies that $(G[Y_x],\tau)$ has a positive dipath from x to u''. Let $P_x'' = xe_1x_1 \cdots e_{t-1}x_{t-1}e_tx_t$ $(x_t = u'')$ be a positive dipath from x to u'' in $(G[Y_x],\tau)$. Then $t_x \notin E(P_x'')$ since t_x is a sink edge. If $E(P_x') \cap E(P_x'') = \emptyset$, then $P_x' + t_x + P_x''$ is a tailless tadpole with tail end x.

If $E(P_x') \cap E(P_x'') \neq \emptyset$, then let s be the maximum index in $\{1,2,\ldots,t\}$ such that $e_s \in E(P_x')$. If both P_x' and P_x'' traverse e_s in the same direction, then $P_x' + t_x + P_x''(x_s,u'')$ is a tadpole with tail end x, where $P_x''(x_s,u'')$ is the segment of P_x'' from x_s to u''.

If P_x'' traverses e_s in the opposite direction from P_x', then the segment $P_x''(x,x_s)$ is a negative dipath from x to x_s since e_s is a positive edge. Since $P_x''(x,x_s)$ is a negative dipath, there is a segment $P_x''(x_i,x_j)$ of $P_x''(x,x_s)$ such that $P_x''(x_i,x_j)$ contains an odd number of negative edges and $V(P_x''(x_i,x_j)) \cap V(P_x'(x,x_{s-1})) = \{x_i,x_j\}$. We choose such a segment that i is as small as possible. By the minimality of i, we have that $P_x''(x_i,x_j)$ is a negative dipath from x_i to x_j and the segment $P_x''(x,x_i)$ is a positive dipath from x to x_i. Denote the segment of P_x' from x to x_{s-1} by $P_x'(x,x_{s-1}) = y_0y_1\ldots y_p$, where $y_0 = x$ and $y_p = x_{s-1}$. Then $x_i = y_a$ and $x_j = y_b$ for some $a,b \in \{0,\ldots,p\}$. If $a < b$, then $P_x''(x_i,x_j) + P_x'(x_i,x_j)$ is a closed negative ditrail from $y_a(= x_i)$ to y_a, and thus $P_x'(x,x_i) + P_x''(x_i,x_j) + P_x'(x_i,x_j)$ is a tadpole with tail end x. If $a > b$, then $P_x''(x_i,x_j) + P_x'(x_j,x_i)$ is a closed negative ditrail from $y_b(= x_j)$ to y_b since $P_x'(x_i,x_j)$ is an all-positive dipath from y_b to y_a, and thus $P_x'(x,x_j) + P_x''(x_i,x_j) + P_x'(x_j,x_x)$ is a tadpole with tail end x. This completes the proof of the claim. ∎

By Claim 5, let $P_x + C_x$ be a tadpole with tail end x in $(G[Y_x],\tau)$. Here, P_x is an all-positive dipath from x to a vertex, denoted by y_x; C_x is a closed negative ditrail from y_x to y_x and $V(P_x)\cap V(C_x)=\{y_x\}$. Note that it is possible that P_x is the single vertex x.

Claim 6. *$\partial(\tau,f)(x)=k$, and if $y_x\neq x$, then $\partial(\tau,f)(y_x)=0$.*

Proof. Suppose to the contrary $\partial(\tau,f)(x)\neq k$. Then $\partial(\tau,f)(x)\geq 2k$, since x is a source vertex and $\partial(\tau,f)(x)=\mu k$ for some positive integer μ.

If $\partial(\tau,f)(y_x)=0$, then $y_x\neq x$, so $|E(P_x)|\geq 1$. We can check easily that the new triple $(G,\tau_{\widetilde{E(P_x)}},f_{\widetilde{E(P_x)}})$ satisfies (S1)–(S3), and the set of source vertices is $X\cup\{y_x\}$, a contradiction to (S4).

If $\partial(\tau,f)(y_x)\neq 0$, since P_x+C_x is a negative ditrail from x to y_x, the new triple $(G,\tau_{\widetilde{E'}},f_{\widetilde{E'}})$ (where $E'=E(P_x+C_x)$) satisfies (S1) and (S2). However, the total sum of boundaries is reduced by $2k$. This contradicts (S3), and so the claim holds. Therefore $\partial(\tau,f)(x)=k$.

Now assume $y_x\neq x$. Since P_x+C_x is a negative ditrail from x to y_x, by Claim 3, $y_x\notin X$, and thus $\partial(\tau,f)(y_x)=0$. ∎

For the sake of convenience, let $(G,\tau_{\widetilde{E(P_x)}},f_{\widetilde{E(P_x)}})=(G,\tau_x,f_x)$, and let X' be the set of source vertices of (τ_x,f_x). The next two claims show that (G,τ_x,f_x) has the same properties as (G,τ,f) and will replace (G,τ,f) in the rest of the proof to obtain a contradiction.

Claim 7. *The following statements for (G,τ_x,f_x) are true.*
(a) *C_x is a tailless tadpole with tail end y_x in (G,τ_x);*
(b) *$X'=(X\setminus\{x\})\cup\{y_x\}$;*
(c) *(G,τ_x,f_x) satisfies (S1)–(S4).*

Proof. The statement (a) is trivial since $E(C_x)\cap E(P_x)=\emptyset$ and C_x is a tailless tadpole with tail end y_x in (G,τ). Now we show the statements (b) and (c). In fact, if $y_x=x$, then $X'=X$ and $(\tau_x,f_x)=(\tau,f)$, and thus both (b) and (c) are trivial; if $y_x\neq x$, then by Claim 6, we can also check directly that both (b) and (c) hold. ∎

Similar to Claims 1 and 3, it follows from Claim 7(c) that (τ_x,f) contains no sink vertex and (G,τ_x) contains no negative ditrail between two distinct vertices of X'.

The next claim basically tells that for any two distinct vertices $x_1,x_2\in X$ (if any), $Y_{x'}\cap V(C_x)=\emptyset$. It will be applied to show that there is exactly one source vertex.

Claim 8. *For every $x'\in X'\setminus\{y_x\}$, (G,τ_x) contains no dipath from x' to C_x.*

Proof. Suppose to the contrary that P is a dipath from x' to y with $V(P)\cap V(C_x)=\{y\}$ in (G,τ_x). Since C_x is a closed negative ditrail from y_x to y_x (by Claim 7(a)) and $y\in V(C_x)$, C_x can be decomposed into two edge-disjoint ditrails from y_x to y, denoted by C_1 and C_2. Since C_x is negative, one of C_1 and C_2 is positive and the other one is negative. Thus either $P+C_1$ or $P+C_2$ is a negative dipath from x' to y_x. This contradicts that (G,τ_x) contains no negative ditrails between two distinct vertices of X'. ∎

Claim 9. $X=\{x\}$.

Proof. Suppose to the contrary $x'\in X\setminus\{x\}$. Then $x'\in X'\setminus\{y_x\}$ by Claim 7(b). Let

$$Y_{x'}=\{y\in V(G):(G,\tau_x)\text{ contains a dipath from }x'\text{ to }y\}.$$

By Claim 8, $Y_{x'}\cap V(C_x)=\emptyset$. Note that (G,τ_x,f_x) satisfies (S1)–(S4) by Claim 7(c). Similar to the discussion in Claims 4 and 5, $(G[Y_{x'}],\tau_x)$ contains a tadpole with tail end

x'. By the definition, there is an unbalanced circuit, denoted by $C_{x'}$, in this tadpole. Since (G,σ) contains no long barbells, $V(C_x) \cap V(C_{x'}) \neq \emptyset$, so $Y_{x'} \cap V(C_x) \neq \emptyset$. This contradicts $Y_{x'} \cap V(C_x) = \emptyset$. ◻

Now we can complete the proof.

The final step. By Claim 9, $X = \{x\}$. By Claim 6, $\partial(\tau, f)(x) = k$ which is an odd number. Since the boundary of every negative edge is an even number, the total sum of the boundaries of (τ, f) on $V(G) \cup E(G)$ must be odd since x is the only source/sink vertex with an odd boundary. This contradicts Observation 3.9. Hence the proof of Theorem 3.5 is complete. ◻

There are precisely two abelian groups of order 4, namely, the Klein Four Group $\mathbb{K}_4$ and the cyclic group $\mathbb{Z}_4$. Clearly, the elements of the Klein Four Group are self-inverse and therefore, a signed cubic graph G has a nowhere-zero $\mathbb{K}_4$-flow if and only if the underlying unsigned graph of G is 3-edge-colorable. We will show that this is also true for signed graphs without long barbells which admit a nowhere-zero $\mathbb{Z}_4$-flow. We will apply a result of Mačajova and Škoviera. A signed graph (G,σ) is *antibalanced* if it is equivalent to a signed graph (G,σ') with $E_N(G,\sigma') = E(G)$.

THEOREM 3.12 (Máčajová and Škoviera [12]). *A signed cubic graph admits a nowhere-zero $\mathbb{Z}_4$-flow if and only if it admits an antibalanced 2-factor.*

THEOREM 3.13. *Let (G,σ) be a flow-admissible signed cubic graph. If (G,σ) contains no long barbells, then (G,σ) admits a nowhere-zero $\mathbb{Z}_4$-flow if and only if the underlying unsigned graph G is 3-edge-colorable.*

Proof. First assume that (G,σ) admits a nowhere-zero $\mathbb{Z}_4$-flow. By Theorem 3.12, (G,σ) has an antibalanced 2-factor $\mathcal{F}$. Since (G,σ) contains no long barbells and $\sum_{C\in\mathcal{F}} |V(C)| = |V(G)| \equiv 0 \pmod 2$, it follows that that every circuit of $\mathcal{F}$ is of even length, so G is 3-edge-colorable.

Now assume that G is 3-edge-colorable. Then $E(G)$ can be decomposed into three edge-disjoint 1-factors M_1, M_2, and M_3. Without loss of generality, assume $|M_1 \cap E_N(G,\sigma)| \equiv |M_2 \cap E_N(G,\sigma)| \pmod 2$. Let $C = M_1 \cup M_2$. Clearly, C is a 2-factor of G.

Since $|E(C) \cap E_N(G,\sigma)| = |M_1 \cap E_N(G,\sigma)| + |M_2 \cap E_N(G,\sigma)| \equiv 0 \pmod 2$, C contains an even number n of unbalanced circuits. Since (G,σ) contains no long barbells, it follows $n = 0$. This implies that each component of C is a balanced circuit with even length and thus is antibalanced. By Theorem 3.12, (G,σ) admits a nowhere-zero $\mathbb{Z}_4$-flow. ◻

Theorem 3.4 doesn't hold for $k = 4$. There is a signed W_5 (the wheel with six vertices) which has a nowhere-zero $\mathbb{Z}_4$-flow but doesn't have a nowhere-zero 4-flow (see [3]).

However, we don't know whether Theorem 3.5 can be extended to all even positive integers $k \geq 6$. We conclude this section with the following problem.

PROBLEM 3.14. *Let $k \geq 6$ be an even integer and (G,σ) be a signed graph with a nowhere-zero $\mathbb{Z}_k$-flow (τ, f_1). If (G,σ) contains no long barbells, does there exist a nowhere-zero k-flow (τ, f_2) such that*

$$f_1(e) \equiv f_2(e) \pmod k?$$

4. Circuit decomposition and sum of 2-flows.

The following theorem is well-known for unsigned graphs.

THEOREM 4.1. *Every Eulerian unsigned graph has a circuit decomposition.*

Theorem 4.1 for unsigned graphs is extended to the class of signed graphs without long barbells.

THEOREM 4.2. *Let (G, σ) be a flow-admissible signed Eulerian graph with $|E_N(G, \sigma)|$ even. If (G, σ) contains no long barbells, then (G, σ) has a decomposition $\mathcal{C}$ such that each member of $\mathcal{C}$ is either a balanced circuit or a short barbell.*

Proof. Suppose to the contrary that (G, σ) is a counterexample. Since (G, σ) is a signed eulerian graph, it has a decomposition $\mathcal{C} = \{C_1, \ldots, C_h, C_{h+1}, \ldots, C_{h+m}, C_{h+m+1}, \ldots, C_{h+m+n}\}$, where h, m, and n are three nonnegative integers, and C_i is an balanced circuit if $i \in \{1, \ldots, h\}$, a short barbell if $i \in \{h+1, \ldots, h+m\}$, and an unbalanced circuit otherwise. We choose such a decomposition that $h+m$ is as large as possible. Then $n \neq 0$. Furthermore, $n \geq 2$ is even since $|E_N(G, \sigma)| \equiv |E_N(C_i, \sigma|_{E(C_i)})| \equiv 0 \pmod 2$ for each $i \in \{1, \ldots, h+m\}$. Since (G, σ) contains no long barbells, it also contains no vertex disjoint unbalanced circuits, and thus, C_{h+m+1} and C_{h+m+2} have at least two common vertices. Let x_1 and x_2 be two common vertices of C_{h+m+1} and C_{h+m+2} such that C_{h+m+1} has a path P_1 from x_1 to x_2 containing no vertex of C_{h+m+2} as internal vertex. Let P_2 and P_3 be the two paths from x_1 to x_2 in C_{h+m+2}. Since C_{h+m+2} is an unbalanced circuit, there is exactly one of P_2 and P_3, say P_2, such that $|E_N(P_1)| \equiv |E_N(P_2)| \pmod 2$, so $P_1 + P_2$ is a balanced circuit of $(G \setminus \cup_{i=1}^{h+m} E(C_i))$. This contradicts the choice of $\mathcal{C}$. ∎

Next we are going to study the decomposition of nowhere-zero k-flows into elementary 2-flows. One of the basic theorems in flow theory for unsigned graphs is Theorem 1.3. The next theorem extends this result to the class of signed graphs without long barbells.

THEOREM 4.3. *Let (G, σ) be a signed graph without long barbells and (τ, f) be a nonnegative k-flow of (G, σ) where $k \geq 2$. Then*

$$(\tau, f) = \sum_{i=1}^{k-1} (\tau, f_i),$$

where each (τ, f_i) is a nonnegative 2-flow.

We need some lemmas to prove Theorem 4.3.

LEMMA 4.4. *Let (G, σ) be a signed graph and (τ, f) be a k-flow of (G, σ). Then the total number of negative edges with odd flow values is even.*

Proof. Denote $F = \{e \in E_N(G, \sigma) : f(e) \text{ is odd}\}$. By Observation 3.9, $\sum_{e \in E_N(G,\sigma)} (-2\tau(h)) f(e) = 0$, and thus $\sum_{e \in E_N(G,\sigma)} \tau(h) f(e) = 0$, where h is a half-edge of e. Therefore $|F| \equiv \sum_{e \in F} \tau(h) f(e) \equiv 0 \pmod 2$. ∎

THEOREM 4.5 (Xu and Zhang [26]). *A signed graph (G, σ) admits a nowhere-zero 2-flow if and only if each component of (G, σ) is Eulerian and has an even number of negative edges.*

LEMMA 4.6. *Let (G, σ) be a signed graph without long barbells and (τ, f) be a k-flow of (G, σ). Let $(Q, \sigma|_{E(Q)})$ be the subgraph of (G, σ) induced by the edges of $\{e : f(e) \equiv 1 \pmod 2\}$. Then every component of $(Q, \sigma|_{E(Q)})$ has an even number of negative edges, and thus $(Q, \sigma|_{E(Q)})$ admits a nowhere-zero 2-flow.*

Proof. Obviously, $(Q, \sigma|_{E(Q)})$ is an even subgraph of (G, σ). By Lemma 4.4, $(Q, \sigma|_{E(Q)})$ has an even number of negative edges and thus the number of components of $(Q, \sigma|_{E(Q)})$ with an odd number of negative edges is even. By Theorem 4.5, if a component of $(Q, \sigma|_{E(Q)})$ has an odd number of negative edges, then it is unbalanced. Thus $(Q, \sigma|_{E(Q)})$ has an even number of unbalanced components. Since (G, σ) contains no long barbells, $(Q, \sigma|_{E(Q)})$ does not contain two vertex-disjoint unbalanced circuits. Therefore, each component of $(Q, \sigma|_{E(Q)})$ is balanced, and thus by Theorem 4.5 again, it admits a nowhere-zero 2-flow. ∎

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Prove by induction on k. It is trivial if $k = 2$. Now assume that the theorem is true for all $t \leq k-1$. Let (τ, f) be a nonnegative k-flow of (G, σ). For convenience, every flow is a flow of (G, σ) under the orientation τ in the following.

We first consider the case when k is odd. Let $(Q, \sigma|_{E(Q)})$ be the subgraph of (G, σ) induced by the edges of $\{e : f(e) \equiv 1 \pmod 2\}$. By Lemma 4.6, (G, σ) admits a 2-flow g with $\mathrm{supp}(g) = E(Q)$. Then each

$$g_1 = \frac{f+g}{2}, \quad \text{and} \quad g_2 = \frac{f-g}{2}$$

is a nonnegative $(\frac{k-1}{2} + 1)$-flow. By induction hypothesis, each g_i is the sum of $\frac{k-1}{2}$ nonnegative 2-flows. Thus $f = g_1 + g_2$ is the sum of $k-1$ nonnegative 2-flows.

Now assume that k is even. Then $k-1$ is odd. First consider f as a modulo $(k-1)$-flow. Then by Theorem 3.5, (G, τ) has a $(k-1)$-flow g satisfying the following two properties:

(a) $f(e) \equiv g(e) \pmod{k-1}$ for each edge $e \in E(G)$;

(b) $\mathrm{supp}(g) = \mathrm{supp}(f) \setminus \{e \in E(G) : f(e) = k-1\}$.

Now in the rest of the proof, we consider f as an integer k-flow. Since $0 \leq f(e) \leq k-1$ and $-(k-2) \leq g(e) \leq k-2$, for each edge e we have $-(k-2) \leq f(e) - g(e) \leq 2k-3$. Thus we have the following properties for $f - g$:

- $(f-g)(e) = 0$, or $k-1$ by (a);
- $\{e \in E(G) : f(e) = k-1\} \subseteq \mathrm{supp}(f-g)$ by (b).

Note that by (b), for each edge e, if $f(e) = 0$, then $g(e) = 0$. Thus $f_1 = \frac{f-g}{k-1}$ is a nonnegative 2-flow with $\{e \in E(G) : f(e) = k-1\} \subseteq \mathrm{supp}(f_1)$. Therefore $f - f_1$ is a nonnegative $(k-1)$-flow. By induction hypothesis, $f - f_1$ is the sum of $k-2$ nonnegative 2-flows. Together with f_1, f can be expressed as the sum of $k-1$ nonnegative 2-flows. This completes the proof of the theorem. ∎

5. Integer and circular flow numbers.

As mentioned in the introduction, $\Phi_i(H) = \lceil \Phi_c(H) \rceil$ holds for each unsigned graph H (Goddyn et al. [2]) but there are signed graphs with $\Phi_i(G, \sigma) - \Phi_c(G, \sigma) \geq 1$. In this section we study the circular flow numbers of signed graphs and prove that signed graphs without long barbells behave like unsigned graphs in this context.

Most examples with the property $\lceil \Phi_c(G, \sigma) \rceil < \Phi_i(G, \sigma)$ contain a star-cut. A star-cut is an induced subgraph S isormorphic to $K_{1,t}$ of G such that every edge of S is an edge-cut of G. It becomes natural to ask whether for each 2-edge-connected signed graph (G, σ) the numbers $\lceil \Phi_c(G, \sigma) \rceil$ and $\Phi_i(G, \sigma)$ are the same. We present an infinite family of counterexamples to this questions. Kompišová and Máčajová [10] present a family of bridgeless cubic signed graphs which also are counterexamples to this question.

Proposition 5.1. *Let t be a positive integer and G_t be the unsigned graph obtained by identifying t copies of K_4 at a common edge v_1v_2. Let (G, σ) be the signed* *graph obtained from G_t by deleting v_1v_2 and adding two negative loops L_1, L_2 at v_1 and v_2, respectively. Then $\Phi_c(G,\sigma) \leq 3$ and $\Phi_i(G,\sigma) \geq 4$.*

Proof. Note that it is easy to check that the unsigned graph G_t does not admit a nowhere-zero 3-flow but admits a positive nowhere-zero 4-flow (D, f) with precisely one edge v_1v_2 with flow value 3.

We first claim that (G,σ) admits a circular nowhere-zero 3-flow. Assume that v_1v_2 is oriented away from v_1 and toward v_2 in D. Orient L_1 away from v_1 and orient L_2 toward v_2 and define a mapping ϕ on $E(G)$ from f by $\phi(e) = f(e)$ for each $e \notin \{L_1, L_2\}$ and $\phi(L_1) = \phi(L_2) = 1.5$. Then ϕ is a circular 3-flow of (G,σ), so $\Phi_c(G,\sigma) \leq 3$.

Now we claim that (G,σ) does not admit a nowhere-zero 3-flow. Suppose to the contrary that (G,σ) admits a nowhere-zero 3-flow and thus admits a nowhere-zero $\mathbb{Z}_3$-flow (τ, g) such that $g(e) = 1$ for every $e \in E(G)$. Since every vertex in $V(G)\setminus\{v_1, v_2\}$ is of degree three in G, every copy of $K_4 - v_1v_2$ contributes zero to $\partial(\tau, g)(v_i)$ for each $i \in \{1, 2\}$. Thus $|\partial(\tau, g)(v_i)| = 2|g(L_i)| \not\equiv 0 \pmod 3$, a contradiction. ∎

The following structural lemma is needed in the proofs of Theorems 5.4 and 5.6. Given a circular $(\frac{p}{q}+1)$-flow (τ,ψ) of a signed graph (G,σ), let $F_\psi = \{e \in E(G) : q\psi(e) \notin \mathbb{Z}\}$.

Lemma 5.2. *Let (G,σ) be a signed graph admitting a circular $(\frac{p}{q}+1)$-flow, and let (τ,ϕ) be a circular $(\frac{p}{q}+1)$-flow of (G,σ) such that F_ϕ has minimum cardinality. If $F_\phi \neq \emptyset$, then*

(1) *the signed induced graph $(G[F_\phi], \sigma|_{F_\phi})$ consists of a set of vertex-disjoint unbalanced circuits;*
(2) *for every edge $e \in E(G)\setminus F_\phi$, $2q\phi(e)$ is an even integer, while for every edge $e \in F_\phi$, $2q\phi(e)$ is an odd integer.*

Proof. Without loss of generality, we may assume $\phi(e) > 0$ for every edge $e \in E(G)$.

I. $(G[F_\phi], \sigma|_{F_\phi})$ contains no signed circuits.

Suppose to the contrary that $(G[F_\phi], \sigma|_{F_\phi})$ contains a signed circuit C. Then (G,σ) admits an integer 2- or 3-flow (τ,ϕ_1) with $\mathrm{supp}(\phi_1) = E(C)$ (see [1]). Let $\epsilon = \min_{e\in E(C)} \min\{\frac{1}{\phi_1(e)}(\frac{p}{q} - \phi(e)), \frac{1}{\phi_1(e)}(\phi(e) - 1)\}$. Then both $(\tau, \phi + \epsilon\phi_2)$ and $(\tau, \phi - \epsilon\phi_2)$ are circular $(\frac{p}{q}+1)$-flows and at least one of $F_{\phi+\epsilon\phi_2}$ and $F_{\phi-\epsilon\phi_2}$ is a proper subset of F_ϕ, contradicting the choice of ϕ.

II. $G[F_\phi]$ is 2-regular.

It is easy to see that the minimum degree $\delta(G[F_\phi]) \geq 2$ since $(\tau, q\phi)$ is a flow with integer value in $E(G)\setminus F_\phi$ and noninteger value only in F_ϕ.

Suppose that Q is a component of $G[F_\phi]$ with maximum degree $\Delta(Q) \geq 3$. Then Q must contain at least two distinct circuits C_1 and C_2, otherwise Q itself is a circuit. By **I**, both C_1 and C_2 are unbalanced. Hence, one may find either a balanced circuit or a short barbell if C_1 and C_2 intersect each other, or a long barbell if C_1 and C_2 are vertex-disjoint, contradicting **I**.

Obviously, (1) is a corollary of **I** and **II**. To prove (2), let $e \in E(G)$. Since $q\phi(e)$ is not an integer if and only if $e \in F_\phi$, $2q\phi(e)$ is an even integer if $e \in E(G)\setminus F_\phi$. Assume $e \in F_\phi$ below. By (1), let C be the unbalanced circuit in $(G[F_\phi], \sigma|_{F_\phi})$ containing e.

Without loss of generality, further assume that e is the unique negative edge of C after switching. Hence, by (1) again,

$$|2q\phi(e)| \equiv |\sum_{v\in V(C)} \partial(\tau, q\phi)(v)| \equiv 0 \pmod 1.$$

Thus $2q\phi(e)$ is an odd integer since $q\phi(e)$ is not an integer. This completes the proof of the lemma. ∎

DEFINITION 5.3. *Let μ be a positive integer. A signed graph (G,σ) is $\frac{1}{\mu q}$-flow-normalizable if it admits a circular $\frac{p}{q}$-flow with rational flow values in $\{1, 1+\frac{1}{\mu q}, 1+\frac{2}{\mu q}, \ldots, \frac{p}{q}-1-\frac{1}{\mu q}, \frac{p}{q}-1\}$ whenever it admits a circular $\frac{p}{q}$-flow with real flow values in $[1, \frac{p}{q}-1]$. By $\mathcal{G}_\mu$ we denote the family of signed graphs which are $\frac{1}{\mu q}$-flow-normalizable.*

For unsigned graphs we have $\mathcal{G}_1 = \mathcal{G}_\mu = \{G : G \text{ is a bridgeless graph}\}$ for each $\mu \geq 2$ (see [21]). However, for general signed graphs this does not hold. As an example we refer to the graph depicted in Figure 5 with $\Phi_c(G,\sigma) = 4$, where it is easy to see that every circular 4-flow must contain an edge with flow value $1+\frac{1}{2}$.

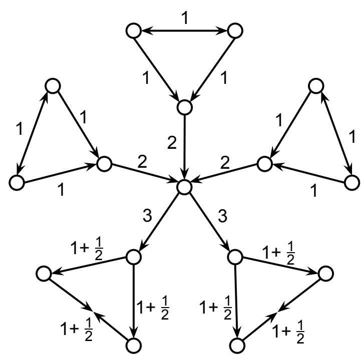

FIG. 5. *A nowhere-zero circular 4-flow of a graph (G,σ) with $\Phi_i(G,\sigma) = 5$.*

The following theorem is a direct corollary of Lemma 5.2(2) and the definition of $\mathcal{G}_2$.

THEOREM 5.4. *A signed graph (G,σ) is flow-admissible if and only if $(G,\sigma) \in \mathcal{G}_2$.*

The following lemma gives some sufficient conditions for $\lceil\Phi_c(G,\sigma)\rceil = \Phi_i(G,\sigma)$.

LEMMA 5.5. *Let $(G,\sigma) \in \mathcal{G}_1$. Then $\lceil\Phi_c(G,\sigma)\rceil = \Phi_i(G,\sigma)$.*

Proof. Let $(G,\sigma) \in \mathcal{G}_1$ with a circular $\frac{p}{q}$-flow (τ, f). Let $k = \lceil\frac{p}{q}\rceil$. Since (τ, f) can also be considered as a circular $\frac{k}{1}$-flow, by Definition 5.3, (G,σ) admits a circular $\frac{k}{1}$-flow (τ, f') with rational flow values in $\{1, 1+\frac{1}{1}, 1+\frac{2}{1}, \ldots, k-1-\frac{1}{1}, k-1\}$. Obviously, (τ, f') is a nowhere-zero k-flow. ∎

THEOREM 5.6. *Let (G,σ) be a signed graph containing no long barbells. Then $(G,\sigma) \in \mathcal{G}_1$, and thus $\lceil\Phi_c(G,\sigma)\rceil = \Phi_i(G,\sigma)$.*

Proof. Suppose that (G,σ) admits a circular $(\frac{p}{q}+1)$-flow. Without loss of generality, assume that G is connected. We choose a circular $(\frac{p}{q}+1)$-flow (τ,ϕ) of (G,σ) such that $F_\phi = \{e \in E(G) : q\phi(e) \notin \mathbb{Z}\}$ has minimum cardinality. If $F_\phi = \emptyset$, then $(G,\sigma) \in \mathcal{G}_1$ by the definition of $\mathcal{G}_1$.

Now assume $F_\phi \neq \emptyset$. Then by Lemma 5.2(1), $G[F_\phi]$ consists of a set of vertex-disjoint unbalanced circuits. Since G is connected and (G,σ) has no long barbells, (G,σ) does not contain two vertex-disjoint unbalanced circuits. Thus $(G[F_\phi], \sigma|_{F_\phi})$ is an unbalanced circuit. By switching, we may assume that $G[F_\phi]$ is an unbalanced circuit with precisely one negative edge, denoted by e_0.

Since (τ,ϕ) is a circular flow of (G,σ), so does $(\tau, q\phi)$. By Observation 3.9, the total sum of the boundaries on $E(G)$ is zero for $(\tau, q\phi)$. By Lemma 5.2(2),

$$0 = \sum_{e\in E(G)} \partial(\tau, q\phi)(e) \equiv \sum_{e\in E_N(G,\sigma)\cap F_\phi} 2q\phi(e) \equiv 2q\phi(e_0) \equiv 1 \pmod 2.$$

This contradiction completes the proof of the theorem. ∎

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