MATH 155 – Review Problems for EXAM 4

1. Indeterminate Forms and L'Hospital's Rule

(1.1) Find the limits.

(a) $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$, (where a and b are positive constants).

(b) $\lim_{x\to 0} \frac{x + \tan^{-1}(x)}{\sin(x)}$.

(c) $\lim_{x\to 0} \frac{e^{3x} - 1}{x}$.

(d) $\lim_{x\to \frac{\pi}{2}} \frac{1 - \sin(x)}{\csc(x)}$.

(e) $\lim_{x\to \infty} \frac{\ln(\ln(x))}{x}$.

(f) $\lim_{x\to 1} \frac{\ln(x)}{\sin(\pi x)}$.

(g) $\lim_{x\to \infty} \frac{e^x}{x^2}$.

(h) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$.

(i) $\lim_{x\to -\infty} x^2 e^x$.

(j) $\lim_{x\to 0^+} \sin(x)\ln(x)$.

(k) $\lim_{x\to \infty} x\tan(\frac{1}{x})$.

(l) $\lim_{x\to 0}(\csc(x) - \cot(x))$.

(m) $\lim_{x\to 1} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$.

(n) $\lim_{x\to 0^+} (\tan(2x))^x$.

(o) $\lim_{x\to \infty} \left(1 + \frac{10}{x}\right)^{3x}$.

(p) $\lim_{x\to \infty} \left(1 + \frac{a}{x}\right)^{bx}$, (where a and b are positive constants).

(q) $\lim_{x\to \infty} x^{\ln(2)/(1+\ln(x))}$.

2. Maximum and Minimum Values

Useful Facts and Tools

Extremal Value Theorem: If f is continuous on a closed interval $[a, b]$, then for some points c and d inside $[a, b]$, f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$.

Critical number: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exists.

How to find local extremities: If f has a local maximum or local minimum at a point c, then c must be a critical number of f.

The Close Interval Method is used to find absolute maximum/minimum values of a continuous function f over a closed interval $[a, b]$:
Step 1: Find all the critical numbers $c_1, c_2, ...$ of f inside $[a, b]$.
Step 2: Determine the absolute maximum/minimum values of f by comparing the values $f(a), f(c_1), f(c_2), ...f(b)$.

(2.1) Find the critical numbers of these functions:
(a) $f(x) = x^3 + x^2 - x$.
(b) $f(x) = x^3 + x^2 + x$.
(c) $f(x) = |3x - 4|$.
(d) $f(x) = (x^2 - x)^{\frac{1}{3}}$.
(e) $f(x) = 4x - \tan(x)$.
(f) $f(x) = xe^{2x}$.

(2.2) Find the absolute maximum and absolute minimum values of each function below on the given interval.
(a) $f(x) = x^3 - 3x + 1$ on $[0, 3]$.
(b) $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1, 4]$.
(c) $f(x) = (x^2 - 1)^3$ on $[-1, 2]$.
(d) $f(x) = x\sqrt{4 - x^2}$ on $[-1, 2]$.
(e) $f(x) = \dfrac{x}{x^2 + 4}$ on $[0, 3]$.
(f) $f(x) = x - 2\cos(x)$ on $[-\pi, \pi]$.
(g) $f(x) = x - \ln(x)$ on $[\frac{1}{2}, 2]$.
(h) $f(x) = x^a(1 - x)^b$ on $[0, 1]$, where $a > 0$ and $b > 0$ are constants.

3. The Mean Value Theorem

Useful Facts and Tools

Mean Value Theorem: If f is continuous on $[a, b]$ and $f'(x)$ exists for any x inside (a, b), then there must a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

When $f(a) = f(b)$, this gives the Rolle's Theorem.

Functions with the same derivatives: If f, g are differentiable functions on an interval I and if $f'(x) = g'(x)$, then for some constant C, $f(x) = g(x) + C$.

(3.1)
(a) Explain why $f(x) = x^3+x-1$ is continuous on $[0,2]$ and differentiable on $(0,2)$, (thus it satisfies the hypothesis of the Mean Value Theorem), find all number c that satisfy the conclusion of the Mean Value Theorem.
(b) Explain why $f(x) = \frac{x}{x+2}$ is continuous on $[1,4]$ and differentiable on $(1,4)$, (thus it satisfies the hypothesis of the Mean Value Theorem), find all number c that satisfy the conclusion of the Mean Value Theorem.

(3.2)
(a) Explain why the equation $2x - 1 = \sin(x)$ has exactly one solution. (Hint: If it has more than one solutions, then how does Rolle's Theorem would tell us?)
(b) Let c denote a constant. Explain why the equation $x^4 + 4x - c = 0$ has at most two solutions. (Hint: If it has more than two solutions, then how does Rolle's Theorem would tell us?)

(3.3) Suppose that f is differentiable over the real numbers, and if $3 \leq f'(x) \leq 5$ for all x. Explain why $18 \leq f(8) - f(2) \leq 30$?

4. Derivatives and the shape of graphs

Useful Facts and Tools

Increasing/Decreasing Test: If $f' > 0$ ($f < 0$, respectively) on an interval, then f is increasing (decreasing, respectively) on that interval.

Concavity Test: If $f'' > 0$ ($f'' < 0$, respectively) on an interval, then f is concave upward (concave downward, respectively) on that interval.

The shape of the graph near critical numbers can be used to detect local maximum/minimum of a function.

(4.1) For each function f below, find (i) intervals on which f is increasing or decreasing, (ii) local maximum/minimum values, (iii) intervals on which f is concave upward/concave downward and inflection points.

(a) $f(x) = x^4 - 4x - 1$.
(b) $f(x) = \dfrac{x^2}{x^2+3}$.
(c) $f(x) = x^2e^x$.
(d) $f(x) = x\ln(x)$.
(e) $f(x) = 2 + 3x - x^3$.
(f) $f(x) = 200 + 8x^3 + x^4$.
(h) $f(x) = 3x^{\frac{2}{3}} - x$.
(g) $f(x) = \ln(x^4 + 27)$.

(4.2) Find all critical numbers of $f(x) = x^4(x-1)^3$, and determine if f attains a local maximum/minimum at these critical numbers.

(4.3) Sketch the graph of a function f with the given properties.
(a) (Exercise 18, Page 218) $f(1) = f(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point at $(0, 1)$.

(b) (Exercise 20, Page 218) $f'(x) > 0$, if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$, $f'(2) = -0$, $\lim x \to \infty f(x) = 1$, $f(-x) = -f(x)$, $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$.
(4.4) For each function f below, find (i) intervals on which f is increasing or decreasing, (ii) local maximum/minimum values, (iii) intervals on which f is concave upward/concave downward and inflection points, (iv) vertical/horizontal asymptotes.

(a) $f(x) = \dfrac{x^2}{(x-2)^2}$.
(b) $f(x) = x\tan(x)$, on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

5. Curve Sketching

(5.1) Sketch these curves (From Exercises 1-44, Page 225)

(a) $f(x) = 8x^2 - x^4$.
(b) $f(x) = \dfrac{x}{(x-1)^2}$.
(c) $f(x) = \dfrac{x}{x^2-9}$.
(d) $f(x) = \dfrac{x^2}{x^2+9}$.
(e) $f(x) = \dfrac{x^2+1}{x^2-1}$.
(f) $f(x) = s\sqrt{2-x^2}$.
(g) $f(x) = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$.
(h) $f(x) = e^{2x} - e^x$.
(i) $f(x) = x\ln(x)$.

6. Optimization Problems

(6.1) Find a positive number such that the sum of the number and its reciprocal is as small as possible.

(6.2) Find the dimensions of a rectangle with area 10000 m^2 whose perimeter is as small as possible.

(6.3) A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimizes the materials used.

(6.4) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the curve $y = 8 - x^2$.

(6.5) A fence 8 ft tall runs parallel to a tall building at a distance of 4ft from the building. What is the length of the shortest ladder that will reach from the ground over the face to the wall of the building?

(6.6) Boat A leaves a dock at 2:00PM and travels due south at a speed 20km/h. Boat B has been heading due east at 15 km/h and reaches the same dock at 3:00PM the same day. At what time were the two boats closest together?

7. Antiderivatives

An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. If $F'(x) = f(x)$, then the **most general antiderivative** of $f(x)$ is $F(x) + C$, where C is an arbitrary constant.

(7.1) Find the most general antiderivative of each of the functions below. Check your answer by differentiation.
(a) $f(x) = 1 - x^3 + 12x^5$.
(b) $f(x) = 2x + 3x^{1.7}$.
(c) $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}$.
(d) $f(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$.
(e) $f(x) = 3e^x + 7\sec^2(x)$.
(f) $f(x) = \dfrac{2}{1+x^2} + \dfrac{1}{\sqrt{1-x^2}}$.

(7.2) Find the antiderivative F of f satisfying the given condition.
(a) $f(x) = 5x^4 - 2x^5$ and $F(0) = 4$.
(b) $f(x) = 4 - 3(1 + x^2)^{-1}$ and $F(1) = 0$.

(7.3) Find f.
(a) $f''(x) = 2 + x^3 + x^6$.
(b) $f''(x) = \cos(x)$.
(c) $f'(x) = 2x - \dfrac{3}{x^4}$, $x > 0$ and $f(1) = 3$.
(d) $f''(x) = 2e^x + 3\sin(x)$.