

PART 1. This portion of the exam is to test basic knowledge and calculation skills to a correct solution. The questions are multiple choice with “None of these” as a possible valid choice. No partial credit is given in this section, so work very carefully. *Value: 4 points each*

1. The expression $\frac{4}{\sqrt[3]{x^2}}$, when written in exponential form, is

(A) $\frac{4x^2}{3}$ (B) $4x^{-\frac{3}{2}}$ (C) $-4x^{\frac{2}{3}}$ (D) $4x^{-\frac{2}{3}}$

(E) $4^{-1}x^{-\frac{3}{2}}$ (AB) $\frac{4x^{-2}}{3}$ (AC) $-4^{-1}x^{-\frac{2}{3}}$ (AD) None of these.

Solution: With fractional exponents, $\sqrt[3]{x^2} = x^{\frac{2}{3}}$. Since this is in the denominator, we need to use negative exponents and so the answer is $4x^{-\frac{2}{3}}$.

2. The radian measure $\frac{2\pi}{3}$, when converted to degree measure, is equal to

(A) $\left(\frac{2\pi}{3}\right)^{\circ}$ (B) $\left(\frac{3}{8\pi}\right)^{\circ}$ (C) 30°

(D) 120° (E) 135° (AB) None of These.

Solution: $\frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{180^{\circ}}{\pi} = 120^{\circ}$.

3. The limit $\lim_{x \rightarrow 0} \frac{4x}{\sin(x)}$ equals:

(A) $\frac{1}{4}$ (B) 4 (C) DNE (D) 0 (E) 1 (AB) None of these.

Solution: Apply Properties of limits to get

$$\lim_{x \rightarrow 0} \frac{4x}{\sin(x)} = \lim_{x \rightarrow 0} \frac{4}{\frac{\sin(x)}{x}} = \frac{4}{\lim_{x \rightarrow 0} \frac{\sin(x)}{x}} = \frac{4}{1} = 4.$$

4. The solutions of the inequality $x^2 - 3x - 10 < 0$ are
- (A) $\{x \in \mathbf{R} \mid -5 < x < 2\}$ (B) $\{x \in \mathbf{R} \mid x < -10\}$ (C) $\{x \in \mathbf{R} \mid x > -10\}$
- (D) $\{x \in \mathbf{R} \mid -2 < x < 5\}$ (E) $\{x \in \mathbf{R} \mid x > 5\}$ (AB) $\{x \in \mathbf{R} \mid x < -2\}$
- (AC) $\{x \in \mathbf{R} \mid x < -2 \text{ or } x > 5\}$ (AD) None of these.

Solution: Factor $x^2 - 3x - 10 = (x+2)(x-5)$ to see that the parabola $y = x^2 - 3x - 10$ has an upward opening which crosses the x -axis at $x = -2$ and $x = 5$. Therefore, the y -coordinates of the parabola $y = x^2 - 3x - 10$ is less than zero (below the x -axis) if and only if $-2 < x < 5$, and so the solutions of $x^2 - 3x - 10 < 0$ are the points in $\{x \in \mathbf{R} \mid -2 < x < 5\}$.

5. The limit $\lim_{x \rightarrow 2} \frac{2(x^2 - 2x - 3)}{x^2 - 6}$ equals:
- (A) 2 (B) -1 (C) DNE (D) $\frac{2}{3}$
- (E) $\frac{-2}{3}$ (AB) $\frac{-3}{2}$ (AC) 3 (AD) None of these.

Solution: Apply Properties of limits to get

$$\lim_{x \rightarrow 2} \frac{2(x^2 - 2x - 3)}{x^2 - 6} = \frac{\lim_{x \rightarrow 2} 2(x^2 - 2x - 3)}{\lim_{x \rightarrow 2} x^2 - 6} = \frac{2(2^2 - 2 \cdot 2 - 3)}{2^2 - 6} = \frac{2 \cdot (-3)}{-2} = 3.$$

6. The domain of the function $f(x) = \frac{1}{\sqrt{4 - x^2}}$ is
- (A) $\{x \in \mathbf{R} \mid x > 0\}$ (B) $\{x \in \mathbf{R} \mid x > 4\}$ (C) $\{x \in \mathbf{R} \mid -4 < x < 4\}$
- (D) $\{x \in \mathbf{R} \mid -2 < x < 2\}$ (E) $\{x \in \mathbf{R} \mid -4 \leq x \leq 4\}$ (AB) $\{x \in \mathbf{R} \mid -2 \leq x \leq 2\}$
- (AC) $\{x \in \mathbf{R} \mid x < -2 \text{ or } x > 2\}$ (AD) None of these.

Solution: As whatever inside the square root cannot be negative, we must have $4 - x^2 \geq 0$; as whatever in the denominator cannot be zero, we must have $4 - x^2 \neq 0$. Combine such considerations to conclude that the domain of the given function must be the points satisfying $4 - x^2 > 0$.

Note that the parabola $y = 4 - x^2$ has a downward opening which crosses the x -axis at $x = -2$ and $x = 2$. Therefore, the y -coordinates of the parabola $y = 4 - x^2$ is bigger than zero (above the x -axis) if and only if $-2 < x < 2$, and so the domain of given function are the points in $\{x \in \mathbf{R} \mid -2 < x < 2\}$.

7. The limit $\lim_{x \rightarrow 2} \frac{4(x^2 - x - 2)}{4 - x^2}$ equals:

- (A) $-\frac{3}{4}$ (B) -3 (C) DNE (D) $\frac{3}{4}$
(E) 2 (AB) 4 (AC) 3 (AD) None of these.

Solution: As this is a $\frac{0}{0}$ -type of limit, we must cancel zero factors first. Use $2 - x = -(x - 2)$ to get

$$\lim_{x \rightarrow 2} \frac{4(x^2 - x - 2)}{4 - x^2} = \lim_{x \rightarrow 2} \frac{4(x - 2)(x + 1)}{(2 - x)(2 + x)} = \lim_{x \rightarrow 2} \frac{4(x + 1)}{-(2 + x)} = \frac{4 \cdot (2 + 1)}{-4} = -3.$$

8. A line passing through the point $(2, 3)$ and perpendicular to the line $2y - x = 10$ has equation

- (A) $y - 3 = 2(x - 2)$ (B) $y - 3 = \frac{1}{2}(x - 2)$ (C) $y - 3 = -(x - 2)$
(D) $2y - 3 = 2(x - 2)$ (E) $y - 3 = 2(x - 2) + 10$ (AB) None of These.

Solution: Rewrite $2y - x = 10$ to $y = \frac{1}{2}x + 5$ to see that the slope of this line is $\frac{1}{2}$. Any line perpendicular to this line must have slope -2 . As none of the lines in (A)-(E) has -2 as a slope, the answer must be (AB).

9. The limit $\lim_{x \rightarrow 4^-} \frac{|x - 4|}{x - 4}$ equals:

- (A) $\frac{1}{4}$ (B) 4 (C) DNE (D) 0 (E) -1 (AB) 1 (AC) None of these.

Solution: When $x \rightarrow 4^-$, we have $x < 4$ or $x - 4 < 0$, and so $|x - 4| = -(x - 4)$. Thus

$$\lim_{x \rightarrow 4^-} \frac{|x - 4|}{x - 4} = \lim_{x \rightarrow 4^-} \frac{-(x - 4)}{x - 4} = \lim_{x \rightarrow 4^-} -1 = -1.$$

10. Given the function

$$f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases},$$

which of the following statements is correct?

- (A) $f(x)$ is continuous at $x = 1$.
(B) The discontinuity of $f(x)$ at $x = 1$ can be removed by defining $f(1) = 2$.
 (C) $f(x)$ has a non removable discontinuity at $x = 1$.
(D) The discontinuity of $f(x)$ at $x = 1$ can be removed by defining $f(1) = 3$.
(E) None of above is correct.

Solution: When $x \neq 1$, $f(x)$ is a polynomial in either interval and so $x = 1$ is the only discontinuity of the function. Since

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 3x - 1 = 2 \text{ and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 + 1 = 2,$$

$\lim_{x \rightarrow 1} f(x) = 2$ exists. Thus the discontinuity of $f(x)$ at $x = 1$ can be removed by defining $f(1) = 2$.

PART 2: This portion of the exam will be graded on a partial credit basis. **Answers without supporting work shown on the paper will receive NO credit.**

11. (8 points each) Compute each of the limit below.

(a) $\lim_{x \rightarrow 2^-} \frac{|x - 2|}{x^2 - 4}$.

Solution: When $x \rightarrow 2^-$, we have $x < 2$ or $x - 2 < 0$, and so $|x - 2| = -(x - 2)$. It follows

$$\lim_{x \rightarrow 2^-} \frac{|x - 2|}{x^2 - 4} = \lim_{x \rightarrow 2^-} \frac{-(x - 2)}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2^-} \frac{-1}{x + 2} = \frac{-1}{4}.$$

(b) $\lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x$.

Solution: Use $(a - b)(a + b) = a^2 - b^2$ (view $a = \sqrt{x^2 + x}$ and $b = x$) to get

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} = \lim_{x \rightarrow \infty} \frac{(x^2 + x) - x^2}{\sqrt{x^2 + x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + x} + x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{2}. \end{aligned}$$

(c) $\lim_{x \rightarrow 0} x \cot(x)$.

Solution: Note that $\cot(x) = \frac{\cos(x)}{\sin(x)}$. Apply the limit $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ and Limit Properties to get

$$\lim_{x \rightarrow 0} x \cot(x) = \lim_{x \rightarrow 0} x \frac{\cos(x)}{\sin(x)} = \lim_{x \rightarrow 0} \frac{\cos(x)}{\frac{\sin(x)}{x}} = \frac{\lim_{x \rightarrow 0} \cos(x)}{\lim_{x \rightarrow 0} \frac{\sin(x)}{x}} = \frac{\cos(0)}{1} = 1.$$

12. (6 points) Given an example of a function $f(x)$ such that $\lim_{x \rightarrow 0} f(x)$ exists but $f(0)$ is not defined and such that $f(1)$ is defined but $\lim_{x \rightarrow 1} f(x)$ does not exist. (You may sketch the graph of your example with such properties).

Solution: Two examples of correct solutions are given below. (You can either give the expressions of these functions or the graphs of such functions).

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x < 1 \\ 1 - x & \text{if } x \geq 1 \end{cases},$$

or even simpler,

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ -1 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

13. (12 points) Given a function

$$f(x) = \frac{x^2 - 2x - 3}{x^2 - 9},$$

do each of the following:

- (a) Determine the domain and all discontinuities of $f(x)$.

Solution: The domain: $\{x \in \mathbf{R} | x \neq \pm 3\}$ or $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. As fractional functions are continuous in their domains, the discontinuities of the function are $x = -3$ and $x = 3$.

- (b) Find all horizontal and vertical asymptotes of the graph $y = f(x)$.

Solution: First, we note that when $x \neq \pm 3$,

$$f(x) = \frac{x^2 - 2x - 3}{x^2 - 9} = \frac{(x - 3)(x + 1)}{(x - 3)(x + 3)} = \frac{x + 1}{x + 3}.$$

At $x = -3$, as

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} \frac{x + 1}{x + 3} = +\infty, \quad \lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} \frac{x + 1}{x + 3} = -\infty,$$

$x = -3$ is a vertical asymptote of $y = f(x)$.

At $x = 3$, as

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 2} \frac{x + 1}{x + 3} = \frac{4}{6} = \frac{2}{3},$$

$x = 3$ is not a vertical asymptote, and so $x = -3$ is the only vertical asymptote of $y = f(x)$.

For horizontal asymptote, we compute

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{x^2 - 2x - 3}{x^2 - 9} = \lim_{x \rightarrow \pm\infty} \frac{1 - \frac{2}{x} - \frac{3}{x^2}}{1 - \frac{9}{x^2}} = \frac{1 - 0 - 0}{1 - 0} = 1.$$

Thus $y = 1$ is the only horizontal asymptote of $y = f(x)$.

14. (6 points each)

(a) (For Exam 1B) Find all real solutions of $2\sin^2 x + \sin x - 1 = 0$.

Solution: As $2\sin^2 x + \sin x - 1 = (2\sin x - 1)(\sin x + 1) = 0$, either $2\sin x - 1 = 0$ or $\sin x + 1 = 0$. (One can also use quadratic formula to get solutions $\sin x = 1$ and $\sin x = \frac{1}{2}$). If $2\sin x - 1 = 0$, then $\sin x = \frac{1}{2}$, which has only one solution $x = \frac{\pi}{4}$ in $[0, 2\pi]$; If $\sin x + 1 = 0$, then $\sin x = -1$, which has only one solution $x = \pi$ in $[0, 2\pi]$. Therefore, all real solutions of the equation are

$$\frac{\pi}{4} + 2k\pi, \pi + 2k\pi, \text{ where } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

(For Exam 1A) Find all real solutions of $2\sin^2 x + 3\sin x - 2 = 0$.

Solution: As $2\sin^2 x + 3\sin x - 2 = (2\sin x - 1)(\sin x + 2) = 0$, either $2\sin x - 1 = 0$ or $\sin x + 2 = 0$. (One can also use quadratic formula to get solutions $\sin x = -2$ and $\sin x = \frac{1}{2}$). If $2\sin x - 1 = 0$, then $\sin x = \frac{1}{2}$, which has only one solution $x = \frac{\pi}{4}$ in $[0, 2\pi]$; If $\sin x + 2 = 0$, then $\sin x = -2$, which does not have any real solution. Therefore, all real solutions of the equation are

$$\frac{\pi}{4} + 2k\pi, \text{ where } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

(b) Solve the equation $\ln(x^4) = 8$.

Solution: Apply properties of logarithm to get $x^4 = e^{\ln x^4} = e^8$. Thus $x^2 = e^4$ and so $x = \pm e^2$.

15. (6 points) Use the Intermediate Value Theorem to verify that $f(x) = x^5 - 3x + 1$ must have at least one zero in the interval $[0, 1]$.

Solution: The polynomial $x^5 - 3x + 1$ is continuous on all the real numbers and so $f(x)$ is continuous on $[0, 1]$. As

$$f(0) = 0^5 - 3 \cdot 0 + 1 = 1 > 0 \text{ and } f(1) = 1^5 - 3 \cdot 1 + 1 = -2 < 0,$$

the Intermediate Value Theorem of continuous functions assures that for some point c inside the interval $(0, 1)$, we have $f(c) = 0$, and so $f(x) = x^5 - 3x + 1$ must have at least one zero in the interval $[0, 1]$.