PART 1. This portion of the exam is to test basic knowledge and calculation skills to a correct solution. The questions are multiple choice with "None of these" as a possible valid choice. No partial credit is given in this section, so work very carefully. *Value: 5 points each*

1. The integral $\int_0^1 \left(x^2 - 2x + \frac{2}{3}\right) dx$ equals:

   (A) 1 (B) $\frac{1}{3}$ (C) 0 (D) $\frac{1}{2}$ (E) $\frac{2}{3}$ (F) None of these.

   Solution: $\int_0^1 \left(x^2 - 2x + \frac{2}{3}\right) dx = \left[\frac{x^3}{3} - x^2 + \frac{2x}{3}\right]_0^1 = \left[\frac{1}{3} - 1 + \frac{2}{3}\right] - 0 = 0.$

2. The derivative of $F(x) = \int_2^x \cos(t^2)dt$ equals:

   (A) $-\sin(x^2)$ (B) $-\sin(x^2) \cdot (2x)$ (C) $-\sin(x^2) - 2$

   (D) $\cos(x^2) \cdot (2x)$ (E) $\cos(x^2)$ (F) None of these.

   Solution: Apply the Fundamental Theorem of Calculus to get $F'(x) = \cos(x^2)$.

3. The antiderivative $\int \frac{x^2+1}{x} dx$ equals

   (A) $\frac{x^2}{2} - \ln|x| + C$ (B) $\left(\frac{x^3}{3} + x\right) x^{-1} + C$ (C) $\frac{\frac{x^3}{3} + x}{\frac{x^2}{2}} + C$

   (D) $\frac{x^2}{2} + \ln|x| + C$ (E) $\frac{x^3}{3} + x + C$ (F) None of these.

   Solution: Rewrite $\frac{x^2+1}{x} = x + x^{-1}$. Then compute the antiderivative: $\int \frac{x^2+1}{x} dx = \int (x + x^{-1})dx = \int x dx + \int x^{-1} dx = \frac{x^2}{2} + \ln|x| + C.$

4. The derivative of $F(x) = \int_1^{\sqrt{x}} \sin(t^2)dt$ equals:

   (A) $\cos(x) \cdot \frac{1}{2\sqrt{x}}$ (B) $\sin(x)$ (C) $\cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

   (D) $-\sin(x) \cdot \frac{1}{2\sqrt{x}}$ (E) $\sin(x) \cdot \frac{1}{2\sqrt{x}}$ (F) None of these.

Solution: As the upper bound is a function of x, we introduce $u = u(x) = \sqrt{x}$. Apply the Fundamental Theorem of Calculus and Chain Rule to get $F'(x) = \sin(x) \cdot \dfrac{1}{2\sqrt{x}}$.

5. The sum $\displaystyle\sum_{i=1}^{5}(i+1)(i-1)$ equals :

   (A) 55 (B) 50 (C) 24 (D) 0 (E) 15 (F) None of these.

Solution: $\displaystyle\sum_{i=1}^{5}(i+1)(i-1) = 0+3+8+15+24 = 50$. **Another solution** is to use the summation formulae: $\displaystyle\sum_{i=1}^{5}(i+1)(i-1) = \sum_{i=1}^{5} i^2 - \sum_{i=1}^{5} 1 = \frac{5(5+1)(10+1)}{6} - 5 = 55-5 = 50$.

6. For the function $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ 2 & \text{if } x \geq 2 \end{cases}$, the integral $\displaystyle\int_0^5 f(x)dx$ equals:

   (A) 10 (B) 0 (C) 4 (D) 6 (E) 5 (F) $\frac{9}{2}$ (G) $\frac{11}{2}$ (H) None of these.

Solution: $\displaystyle\int_0^5 f(x)dx = \int_0^2 2xdx + \int_2^5 2dx = 4+6 = 10$. **Another solution** is to compute the area of triangle with base 2 and height 4 (and so area is 4) and the area of the 2 by 3 rectangle (area is 6). Therefore, the integral equals the sum of these areas: $4+6 = 10$.

7. The integral $\displaystyle\int_0^1 \sqrt{x}(x+1)dx$ equals

   (A) $\frac{8}{15}$ (B) $\frac{16}{15}$ (C) $\frac{2}{5}$ (D) $\frac{7}{5}$ (E) $\frac{5}{2}$ (F) None of these.

Solution: $\displaystyle\int_0^1 \sqrt{x}(x+1)dx = \int_0^1 (x^{\frac{3}{2}} + x^{\frac{1}{2}})dx = \left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$.

PART 2: This portion of the exam will be graded on a partial credit basis. **Answers without supporting work shown on the paper will receive NO credit.**

8. (*14 points*) Compute the following antiderivatives.

   (a) $\int x\sqrt{x^2+1}dx$.

   Solution: Set $u = x^2+1$. Then $du = 2xdx$ and so $\int x\sqrt{x^2+1}dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{(x^2+1)^{\frac{3}{2}}}{3} + C$.

   (b) $\int \left(x+\frac{1}{x}\right)^2 dx$.

   Solution: Evaluate the antiderivative by first expanding the square: $\int \left(x+\frac{1}{x}\right)^2 dx = \int (x^2+2+x^{-2})dx = \frac{x^3}{3} + 2x - \frac{1}{x} + C$.

9. (*14 points*) Compute the following antiderivatives.

   (a) $\int \cos^3(x)\sin(x)dx$.

   Solution: Set $u = \cos(x)$. Then $du = -\sin(x)dx$ and so $\int \cos^3(x)\sin(x)dx = -\int u^3du = -\frac{u^4}{4} + C = -\frac{\cos^4}{4} + C$.

   (b) $\int x^2e^{-x^3}dx$.

   Solution: Set $u = -x^3$. Then $du = -3x^2dx$ and so $\int x^2e^{-x^3}dx = \frac{-1}{3}\int e^u du = \frac{-1}{3}e^{-x^3} + C$.

10. (*16 points*) Compute the following integrals.

    (a) $\int_0^1 \frac{1}{\sqrt{5x+4}}dx$.

    Solution: Set $u = 5x+4$. Then $du = 5dx$. When $x = 0$, $u = 4$ and when $x = 1, u = 9$. Thus $\int_0^1 \frac{1}{\sqrt{5x+4}}dx = \frac{1}{5}\int_4^9 u^{-\frac{1}{2}}du = \frac{1}{5}\left[2\sqrt{u}\right]_4^9 = \frac{2}{5}(3-2) = \frac{2}{5}$.

(b) $\int_1^e \frac{\ln(x)}{x} dx$.

Solution: Set $u = \ln(x)$. Then $du = \frac{1}{x}dx$. When $x = 1$, $u = \ln(1) = 0$; when $x = e$, $u = \ln(e) = 1$. Thus $\int_1^e \frac{\ln(x)}{x} dx = \int_0^1 u du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$.

11. (*10 points*) Evaluate the integral $\int_0^2 (x^2 - 1)dx$ by computing the limit of a Riemann sum. (**No credit for solutions not using a Riemann sum**).

Solution: $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and for each $i = 0, 1, 2, \cdots, n$, $x_i = \frac{2i}{n}$.

Form the Riemann sum by using the right end evaluation:

$$
\begin{aligned}
R_n &= \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 - 1 \right] \frac{2}{n} = \frac{2}{n} \sum_{i=1}^{n} \left[\frac{4}{n^2} i^2 - 1 \right] \\
&= \frac{8}{n^3} \sum_{i=1}^{n} i^2 - \frac{2}{n} \sum_{i=1}^{n} 1 = \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{2}{n} \cdot n \\
&= \frac{8n(n+1)(2n+1)}{6n^3} - 2.
\end{aligned}
$$

Therefore, the integral is

$$\int_0^2 (x^2 - 1)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(\frac{8n(n+1)(2n+1)}{6n^3} - 2 \right) = \frac{16}{6} - 2 = \frac{8}{3} - \frac{6}{3} = \frac{2}{3}.$$

12. (*11 points*) Find the maximum possible area of a rectangle with a base that lies on the x-axis and with two upper vertices that lie on the graph of $y = 12 - x^2$.

Solution: Let (x, y) denote the upper right corner vertex of the rectangle. Then the area of the rectangle is $A = (2x) \cdot y$. Since $y = 12 - x^2$, this area is a function of x: $A = A(x) = 2x(12 - x^2) = 24x - 2x^3$. from the description of the problem, $0 \le x \le \sqrt{12}$. Hence what we need to do is to find the absolute maximum of the function $A(x)$ over the closed interval $[0, \sqrt{12}]$.

Compute $A'(x) = 24 - 6x^2 = 6(4 - x^2) = 6(2 - x)(2 + x)$. Then within the interval $[0, \sqrt{12}]$, $A(x)$ has only one critical point $x = 2$. Compare the values

$$A(0) = 0, A(2) = 48 - 16 = 32, A(\sqrt{12}) = 0.$$

We conclude that the maximum possible of the area is 32.