ANOTHER BOUND ON THE CHROMATIC NUMBER OF A GRAPH

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Let $G$ be a simple graph, let $\Delta(G)$ denote the maximum degree of its vertices, and let $\chi(G)$ denote its chromatic number. Brooks' Theorem asserts that $\chi(G) \leq \Delta(G)$, unless $G$ has a component that is a complete graph $K_{\Delta(G)+1}$, or unless $\Delta(G) = 2$ and $G$ has an odd cycle. We show here that this bound can be improved if $G$ does not contain certain types of subgraphs. For instance, if $G$ has no 4-cycle, then $\chi(G) \leq \frac{5}{3}(\Delta(G)+3)$. A different result of a similar nature was recently obtained independently by us, by Borodin and Kostochka, and by Lawrence.

1. Introduction

Let $G$ be a simple graph of maximum degree $\Delta(G) = h$ and chromatic number $\chi(G)$. The basic bound on $\chi(G)$ was given by Brooks [2]:

**Theorem 1.1.** For any graph $G$,

$$\chi(G) \leq \Delta(G) + 1,$$

with equality if and only if either $\Delta(G) = 2$ and $G$ contains an odd cycle, or $G$ contains a clique $K_{\Delta(G)+1}$.

The odd cycle or the clique $K_{\Delta(G)+1}$ of Brooks' Theorem is necessarily a component of $G$.

For graphs with no clique $K_{r+1}$, where $r$ is not too large, Brooks' bound was improved independently by Borodin and Kostochka [1], by Catlin [3], and by Lawrence [5]:

**Theorem 1.2.** For any graph $G$ containing no $K_{r+1}$, where $3 \leq r \leq \Delta(G)$,

$$\chi(G) \leq \frac{r}{r+1} (\Delta(G) + 2).$$

Borodin and Kostochka's result was stated in more general terms, and Mitchem [7] generalized the proof in [3] to independently obtain the more general result of [1].

When we use the terms "clique" or "$K_{r+1}$" in $G$, we mean "complete subgraph", and not necessarily "maximal complete subgraph".
Let \( K_{r+2} - e \) denote the complete subgraph on \( r+2 \) vertices, minus an edge. When we say that \( G \) contains no \( K_{r+2} - e \) as a subgraph, we do not mean "induced subgraph". We mean that \( G \) contains no \( K_{r+2} \) also. A 4-cycle is a cycle on 4 edges.

The main result of this paper is:

**Theorem 1.3.** Let \( G \) be a graph. If \( G \) has no \( K_{r+2} - e \) as a subgraph, for some \( r \geq 3 \), then

\[
\chi(G) \leq \frac{r}{r+1} (\Delta(G) + 3).
\]

If also \( G \) has no 4-cycle, then

\[
\chi(G) \leq \frac{2}{3} (\Delta(G) + 3).
\]

In the proofs of Theorem 1.2 (see [1] or [3]) a decomposition theorem of Lovász [6] was used. Here we develop a variation of it as part of the proof of Theorem 1.3. Other variations are in [4]. Lovász's result:

**Theorem 1.4.** Let \( G \) be a graph and let \( n \) be a natural number. For any partition

\[
h_1 + h_2 + \cdots + h_n = \Delta(G) - (n - 1)
\]

of \( \Delta(G) - (n - 1) \), there is a decomposition of \( V(G) \) into sets \( X_1, X_2, \ldots, X_n \) such that \( \Delta(G[X_i]) \leq h_i \), for \( i = 1, 2, \ldots, n \), where \( G[X_i] \) is the subgraph of \( G \) induced by \( X_i \).

One can obtain Theorem 1.2 from Theorems 1.1 and 1.4 by setting most of the \( h_i \)'s equal to \( r \).

Borodin and Kostochka stated in a recent communication that Kostochka has proved that if the girth of \( G \) is at least \( 2\Delta(G)^2 \), then

\[
\chi(G) \leq \frac{1}{2} (\Delta(G) + 4).
\]

The least value of \( \Delta(G) \) for which Theorem 1.3 improves Theorems 1.1 and 1.2 is \( \Delta(G) = 10 \): the theorem gives \( \chi(G) \leq 8 \) if \( G \) has no 4-cycle. The least value of \( \Delta(G) \) for which Theorem 1.3 improves Theorems 1.1 and 1.2 when \( r \geq 3 \) is \( \Delta(G) = 18 \): if \( G \) has a \( K_4 \) but has no \( K_{r+2} - e \), then Theorem 1.3 (with \( r = 3 \)) gives \( \chi(G) \leq 15 \), while Theorem 1.2 gives only \( \chi(G) \leq 16 \).

We know of no examples that would show that Theorem 1.2 or 1.3 is best possible. Thus, we pose the following question: is there a constant \( c_r > 0 \) (depending only on \( r \)) such that for arbitrarily high values of \( h \) there are graphs \( G \) with \( \Delta(G) = h \), with no \( K_{r+1} \)'s and with \( \chi(G) > c_r \Delta(G) \)?
2. The proof of Theorem 1.3

Consider \( r \) to be fixed by a hypothesis that \( G \) contains no \( K_{r+2} - e \). If \( G \) has no 4-cycle, then set \( r = 2 \). Let \( \Delta(G) \) be denoted by \( h \), and let

\[
n = \left\lceil \frac{h + 2}{r + 1} \right\rceil,
\]

where the brackets denote the greatest integer function. Then we can write

\[
h_1 = h_2 = \cdots = h_{n-1} = r,
\]

and so for some integer \( h_n \) satisfying

\[
r \leq h_n = h + 2 - n(r + 1) + r \leq 2r
\]

we have

\[
h = h_1 + h_2 + \cdots + h_{n-1} + h_n + (n - 2).
\]

For a subset \( X_i \subseteq V(G) \), we can write \( G_i = G[X_i] \) (the subgraph of \( G \) induced by \( X_i \)). We also write \( E(X_i) = E(G[X_i]) \). Define the integer-valued function \( f \) by

\[
f(X_1, X_2, \ldots, X_n) = h_1 \cdot |X_1| + h_2 \cdot |X_2| + \cdots + h_n \cdot |X_n| - |E(X_1)| - |E(X_2)| - \cdots - |E(X_n)|,
\]

where \((X_1, X_2, \ldots, X_n)\) is a decomposition of \( V(G) \). In particular, assume that \((X_1, \ldots, X_n)\) is the decomposition of \( V(G) \) that

(i) maximizes \( f(X_1, \ldots, X_n) \);

(ii) if \( r = 2 \), minimizes the total number of odd cycles in all \( G_i \)'s for which \( h_i = 2 \), such that (i) holds;

(iii) if \( r > 2 \), minimizes the total number of cliques \( K_{r+1} \) in the \( G_i \)'s for which \( h_i = r \), again subject to (i).

Thus, if \( h_n > r \), then \( G_n \) is not relevant to (ii) or (iii). Those odd cycles or cliques counted in (ii) or (iii) are clearly components of the respective subgraphs \( G_i \), and we shall refer to them collectively as Brooks components, in order to use a common name.

By the maximality of \( f \) (i.e., by (i)),

\[
0 \leq f(X_1, \ldots, X_n) - f(X_1 - x, X_2 + x, X_3, \ldots, X_n)
\]

\[
0 \leq f(X_1, \ldots, X_n) - f(X_1 - x, X_2, X_3 + x, \ldots, X_n)
\]

\[
\cdots
\]

\[
0 \leq f(X_1, \ldots, X_n) - f(X_1 - x, X_2, X_3, \ldots, X_n + x).
\]
Then by the definition of $f$,

\[ 0 \leq h_1 |X_1| - h_1(|X_1| - 1) + h_2 |X_2| - h_2(|X_2| + 1) - |E(X_1)| + |E(X_1 - x)| - |E(X_2)| + |E(X_2 + x)| \]

\[ \vdots \]

\[ 0 \leq h_1 |X_1| - h_1(|X_1| - 1) + h_n |X_n| - h_n(|x_n| + 1) - |E(X_1)| + |E(X_1 - x)| - |E(X_n)| + |E(X_n + x)|. \]

Hence, for $i = 2, 3, \ldots, n$,

\[ 0 \leq h_1 - h_i - \deg_{G_i, x} x + \deg_{G(X_i + x)} x, \]

and so for each $i \geq 2$ and for $i = 1$, we have

\[ \deg_{G_i, x} x \leq h_1 - h_i + \deg_{G(X_i + x)} x. \]

We sum both sides of this system of inequalities, letting $i$ run from 1 to $n$, and we obtain

\[ n \deg_{G_i, x} x \leq nh_1 - \sum_{i=1}^{n} h_i + \sum_{i=1}^{n} \deg_{G(X_i + x)} x \]

\[ = nh_1 - (h - n + 2) + \deg_{G, x} x \]

\[ \leq nh_1 + n - 2. \]

Dividing both sides by $n$, we get

\[ \deg_{G_i, x} x \leq h_1 + \frac{n - 2}{n}, \]

and since $\deg_{G_i, x}$ and $h_1$ are integers,

\[ \deg_{G_i, x} x \leq h_1. \]

In a similar manner we can obtain for $x \in X_i$ ($i \leq n$)

\[ \deg_{G_i, x} x \leq h_i. \]

Hence, for $i = 1, 2, \ldots, n$,

\[ \Delta(G_i) \leq h_i. \] (\(^*)

We claim that it suffices to show that $\chi(G) \leq h_i$ for each $i$. Were this so, then

\[ \chi(G) \leq \sum_{i=1}^{n} \chi(G_i) \leq \sum_{i=1}^{n} h_i = h - n + 2 \]

\[ = h - \left[ \frac{h + 2}{r + 1} \right] + 2 \]

\[ \leq h - \frac{h + 2}{r + 1} + \frac{1}{r + 1} + 2 \]

\[ = \frac{r}{r + 1} (h + 3), \]
which is the conclusion of the theorem. Thus, we must show that $\chi(G_i) \leq h_i$ for each $i$. If $h_n > r$, then by (\*), Brooks' Theorem (Theorem 1.1) can be applied to $G_n$, for the hypotheses of Theorem 1.3 preclude Brooks components in $G_n$. Specifically, if $r = 2$, then by hypothesis, $G_n$ contains no 4-cycle, and hence no $K_{h_n+1}(h_n \geq 3)$; if $r > 2$, then $G_n$ contains no $K_{r+2} - e$ and hence, no $K_{r+2} \leq K_{h_n+1}(h_n \geq r+1)$. Therefore, we can restrict our attention to eliminating Brooks components from those subgraphs $G_i$ for which $h_i = r$. Then when Brooks' Theorem is applied to these subgraphs, $\chi(G_i) \leq h_i$ follows.

- Suppose by way of contradiction that $G_i$ contains a Brooks component $C_0$ (an odd cycle if $\Delta(G_i) = r = 2$, a clique $K_{r+1}$ if $\Delta(G_i) = r > 2$), for some $i$. Let $x_0 \in V(C_0)$.

If $x_0$ is adjacent to at least $h_i + 1$ vertices in each set $X_j$ for $j \neq i$, then

$$\deg_{C} x_0 \geq h_i + \sum_{j \neq i} (h_j + 1) = h + 1 > h,$$

contrary to hypothesis. Hence, there is some $j$ such that $x_0$ is adjacent to $h_j$ or fewer vertices in $X_j$.

Notice that as $x_0$ is moved from $X_i$ to $X_j$, the maximality of $f$ is preserved:

$$f(X_1, \ldots, X_n) = f(X_1, \ldots, X_i - x_i, \ldots, X_j + x_j, \ldots, X_n).$$

If $h_i > r$, then $j = n$, and since $(X_1, \ldots, X_n)$ was chosen to minimize the number of Brooks components in $G_1, \ldots, G_{n-1}$ (condition (ii) or (iii)), $x_0$ must lie in a Brooks component in $G_n$. But, as we have already seen, Brooks' Theorem and the hypotheses preclude Brooks components in $G_n, \text{ if } h_n > r$. Therefore, $h_i = r$, although $j$ may still equal $n$.

To avoid violating condition (ii) or (iii), that the number of Brooks components is minimized, subject to (i), the destruction of $C_0$ in $G_i$ must be accompanied by the formation of a Brooks component $C_1$ containing $x_0$, as $x_0$ is moved from $G_i$ to $X_i$ to form $G[X_i + x_0]$. An odd arc $C_0 - x_0$ is left behind in $G_i - x_0$ if $r = 2$, and if $r \geq 3$, then a clique $K_r = C_0 - x_0$ remains.

We repeat this process by picking a vertex $x_i \neq x_0$ in $V(C_i)$ and moving it out of $X_i + x_0$. Another Brooks component $C_2$ is formed, leaving behind an odd arc ($r = 2$) or a clique $K_r(r > 2)$.

Since $G$ is finite, this sequence $C_0, C_1, C_2, \ldots$ of Brooks components in the subgraphs will eventually double back on itself for the first time. A vertex $x_{m-1}$ will be moved into a set $X^*$ of the altered decomposition, where it will be part of a Brooks component $C_m$ in $G[X^* + x_{m-1}]$, and where $C_m$ overlaps $C_k$ for some $k < m$. Then either $C_m - x_{m-1}$ is the odd arc $C_k - x_k$ left behind as some vertex $x_k$ in the sequence

$$x_0, x_1, \ldots, x_k, \ldots, x_{m-1}$$

was moved out ($r = 2$), or $C_m - x_{m-1} - C_k - x_k$ is a clique $K_r(r > 2)$ in $G[X^*]$ formed under similar circumstances. In the first case ($r = 2$), $x_{m-1}, x_k$, and the two
endpoints of the odd arc
\[ C_m - x_{m-1} = C_k - x_k \]
form a 4-cycle, contrary to hypothesis; in the second case \((r > 2)\), \(x_{m-1}, x_k\), and the \(r\)-clique
\[ C_m - x_{m-1} = C_k - x_k \]
form a \(K_{r+2} - e\) in \(G\), again contrary to the hypothesis of the theorem.

Thus, there are no Brooks components in the \(G_i\)'s, and so, as already demonstrated, the conclusions of the theorem follow.

References