Hadwiger’s Conjecture is True for Almost Every Graph

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The contraction clique number $\text{ccl}(G)$ of a graph $G$ is the maximal $r$ for which $G$ has a subcontraction to the complete graph $K^r$. We prove that for $d > 2$, almost every graph of order $n$ satisfies $n/(\log_2 n + 4) - 1 \leq \text{ccl}(G) \leq n/(\log_2 n - d \log_2 n + 2)$. This inequality implies the statement in the title.

1. INTRODUCTION

One of the deepest unsolved problems in graph theory is the following conjecture due to Hadwiger [7]: $\chi(G) = s$ implies $G \cong K^s$. In other words, every $s$-chromatic graph $G$ has a subcontraction to $K^s$, the complete graph of order $s$. In the case $s = 5$, this is equivalent to the four-colour theorem. (For an account of the various results related to Hadwiger’s conjecture the reader is referred to [1, Chapter VII]; the terminology and notation not defined here can also be found in [1].)

The statement in the title would sound rather hollow but for certain recent developments. Hajós conjectured that every $s$-chromatic graph contains a $TK^s$, a topological complete subgraph of order $s$, that is a subdivision of $K^s$. This is clearly stronger than Hadwiger’s conjecture, for a $TK^s$ itself has a contraction to $K^s$, but a graph subcontractable to $K^s$ need not contain a $TK^s$. The Hajós conjecture was disproved recently by Catlin [5], who exhibited counter-examples for $\chi(G) \geq 7$. Shortly after Catlin’s result Erdős and Fajtlowicz [6] showed that almost every graph is a counter-example to the Hajós conjecture. More precisely, define the topological clique number of a graph $G$ as

$$\text{tcl}(G) = \max\{r: G \cong TK^r\}.$$ 

Erdős and Fajtlowicz showed that for almost every graph $G$ of order $n$,

$$\text{tcl}(G) \leq cn^\frac{1}{2},$$

for some absolute constant $c$. Since for every $\varepsilon > 0$ almost every graph satisfies

$$\chi(G) \geq (\frac{1}{2} - \varepsilon)n/\log_2 n,$$

we have that

$$\text{tcl}(G) \leq \chi(G)$$

for almost every graph (for sharp results on $\chi(G)$ see [4]).

Inequality (1) was extended by Bollobás and Catlin [3], who proved that for every $\varepsilon > 0$ almost every graph satisfies

$$(2 - \varepsilon)n^\frac{1}{2} \leq \text{tcl}(G) \leq (2 + \varepsilon)n^\frac{1}{2}$$

and so

$$(\frac{1}{2} - \varepsilon)n^\frac{1}{2}/\log_2 n \leq \chi(G)/\text{tcl}(G).$$

In view of this it is imperative to attack Hadwiger’s conjecture by random graphs, that is to examine whether or not Hadwiger’s conjecture holds for almost every graph. This is

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exactly the task we shall accomplish in this note. More precisely, we shall prove an analogue of (2) for the contraction clique number \(\text{ccl}(G)\) of a graph \(G\), defined as

\[
\text{ccl}(G) = \max\{r: G \supset K^r\}.
\]

2. Random Graphs

Let \(0 < p < 1\) be fixed, and let \(V\) be a set of \(n\) distinguishable vertices. Denote by \(\mathcal{G}(n, P(\text{edge}) = p)\) the discrete probability space consisting of all graphs with vertex set \(V\), in which the probability of a graph of size \(m\) is

\[
p^n(1-p)^{{n \choose 2} - m}.
\]

In other words, the edges of a graph \(G \in \mathcal{G}(n, P(\text{edge}) = p)\) are chosen independently and with probability \(p\). (See [2, Chapter VII] for results concerning this model.)

Given a property \(\mathcal{P}\) of graphs we define the probability of \(\mathcal{P}\) as

\[
P(\mathcal{P}) = P(G \in \mathcal{G}(n, P(\text{edge}) = p): \mathcal{P} \text{ holds for } G).
\]

If \(P(\mathcal{P}) \rightarrow 1\) as \(n \rightarrow \infty\) then the property \(\mathcal{P}\) is said to hold for almost every graph.

In order to make the calculations below a little more pleasant, we shall take \(p = \frac{1}{2}\). The case \(p = \frac{1}{2}\) is in some sense the most natural since this is the model one considers implicitly when one counts the proportion of all graphs having a given property. Indeed, in the model \(\mathcal{G} = \mathcal{G}(n, P(\text{edge}) = \frac{1}{2})\) every graph has the same probability, so the probability of a set \(\mathcal{H} \subset \mathcal{G}\) is exactly \(|\mathcal{H}|/|\mathcal{G}|\). Thus a property \(\mathcal{P}\) holds for almost every graph in \(\mathcal{G}(n, P(\text{edge}) = \frac{1}{2})\) if the number of graphs having \(\mathcal{P}\) is asymptotically equal to the number of all graphs (with vertex set \(V\)).

3. The Contraction Clique Number

Given a graph \(G\) and non-empty disjoint subsets \(V_1, V_2, \ldots, V_s\) of \(V = V(G)\), denote by \(G/(V_1, \ldots, V_s)\) the graph with vertex set \(\{V_1, V_2, \ldots, V_s\}\) in which \(V_i\) is joined to \(V_j\) if \(G\) contains a \(V_i - V_j\) edge. Put

\[
\text{ccl}'(G) = \max\{r: G/(V_1, \ldots, V_s) \supset K^r\text{ for some } V_1, \ldots, V_s\}.
\]

Since the contraction clique number is defined similarly, except with the added restriction on the \(V_i\) that each \(G[V_i]\) is connected,

\[
\text{ccl}(G) \leq \text{ccl}'(G).
\]

We shall give a lower bound for \(\text{ccl}(G)\) and an upper bound for \(\text{ccl}'(G)\) holding for almost every graph. As customary, \(\log_b x\) denotes the logarithm to base \(b\).

**Theorem.** Let \(d > 2\). Then almost every graph \(G \in \mathcal{G}(n, P(\text{edge}) = \frac{1}{2})\) satisfies

\[
n((\log_2 n)^{1/4} + 4)^{-1} \leq \text{ccl}(G) \leq \text{ccl}'(G) \\
\leq n((\log_2 n - d \log_2 \log_2 n)^{1/2} - 1)^{-1}.
\]

**Proof.** (a) We start with a proof of the upper bound on \(\text{ccl}'(G)\). Put \(s = \lceil n((\log_2 n - d \log_2 \log_2 n)^{1/2}) \rceil\). A partition \(\{V_1, V_2, \ldots, V_s\}\) of the vertex set \(V\) is said to be permissible for a graph \(G\) if \(G\) contains a \(V_i - V_j\) edge for every pair \((i, j)\), \(1 \leq i < j \leq s\). The clique \(\text{ccl}'(G) \geq s\) iff the graph \(G\) has a permissible partition. We have to prove that the probability that a graph has a permissible partition tends to 0 as \(n \rightarrow \infty\).
To start with, note rather crudely that there are at most
\[ \frac{n!}{s!} \binom{n}{s-1} < n^n \]  
partitions of \( V \) into \( s \) non-empty sets. The number on the left-hand side of (3) is the number of partitions of \( V \) into \( s \) non-empty ordered sets.

Consider now a fixed partition \( \mathcal{P} = \{V_1, V_2, \ldots, V_s\} \) into non-empty sets. What is the probability that this partition \( \mathcal{P} \) is permissible? Let \( n_1, n_2, \ldots, n_s \) be the number of vertices in the classes. Then the probability that a graph contains no \( V_i - V_j \) edge is \( 2^{-n_i} \).

Hence
\[ P(\mathcal{P} \text{ is permissible}) = \prod (1 - 2^{-n_i}) \leq e^{-\sum 2^{-n_i}}, \]  
where both the product and the sum are taken over all pairs \((i, j)\) with \( 1 \leq i < j \leq s \). We have the following string of elementary inequalities.
\[ \sum 2^{-n_i} \binom{s}{2}^{-1} = (\sum n_i \binom{s}{2})^{-1} = 2^{-(\sum n_i \binom{s}{2})^{-1}} \geq 2^{-n^2/s^2}. \]  
The reader may note that \( \sum n_i n_j \) is exactly the number of edges in the complete \( s \)-partite graph with vertex classes \( V_1, V_2, \ldots, V_s \). The Turán graph \( T_s(n) \) is the unique \( s \)-partite graph with maximal number of edges, and
\[ e(T_s(n)) = \left( \frac{s-1}{2s} + o(1) \right) n^2 \]  
(see [2, p. 71]).

From (4) and (5) we have
\[ P(\mathcal{P} \text{ is permissible}) \leq e^{-\left(\frac{s}{2s}\right)2^{-n^2/s^2}}, \]  
and (3) and (6) imply
\[ P(G \text{ has a permissible partition} = P(\text{ccl}'(G) \geq s) \leq n^s e^{-\left(\frac{s}{2s}\right)2^{-n^2/s^2}} = P_s. \]

Clearly
\[ \log P_s = n \log n - \binom{s}{2} 2^{-n^2/s^2} \leq n \left\{ \log n - \frac{1}{3 \log_2 n} 2^{d \log_2 n} \right\} \leq -\frac{1}{4} n (\log_2 n)^2 \rightarrow -\infty. \]

Hence \( P_s \rightarrow 0 \), proving the required upper bound on \( \text{ccl}'(G) \).

(b) We turn to the proof of the lower bound on \( \text{ccl}(G) \). Put \( k = \lceil (\log n)^t + \frac{1}{2} \rceil \), \( s = \lceil n/(k^2/2) \rceil \) and \( t = \lceil n/(k+2) \rceil \). We shall prove in two steps that \( G \geq K_t^s \) for almost every graph \( G \).

Step 1. Fix a set \( T \) of \( t \) vertices and put \( W = V - T \). Then almost every graph \( G \) contains \( t \) vertex disjoint stars of order \( k + 1 \) whose centres are the \( t \) vertices in \( T \).

Indeed, by a slight extension of Hall's theorem (see [2, p. 56]) if \( G \) does not contain such stars then there is a set \( A \subset T \) for which the vertices in \( A \) have less than \( k |A| \) neighbours in \( W \). Given a set \( A \) with \( a = |A| \) elements, the probability that a vertex in \( W \) is joined to no vertex in \( A \) is \( 2^{-a} \). Hence the probability that the vertices in \( A \) have less than \( ka \) neighbours in \( W \) is at most
\[ \sum_{u < ka} \binom{n-r}{u} 2^{-a(n-t-u)} < n^k 2^{-a(n-t-ka)} \]
\[ \leq n^k 2^{-at} < 2^{-av/2}. \]
Consequently the probability that $G$ fails to contain the desired $t$ stars is at most

$$\sum_{a \leq t} \binom{t}{a} 2^{-at} 2^{\frac{a}{2}} \leq \sum_{a \leq t} (t^{-\frac{1}{2}})^a \leq 2t 2^{-\frac{t}{2}},$$

and this tends to 0.

**Step 2.** Let $V_1, V_2, \ldots, V_t$ be the vertex sets of the stars constructed in Step 1 in almost every graph. Then for almost every graph $G$ there are $V_n, V_{n+1}, \ldots, V_n$ such that $G / \{V_n, V_{n+1}, \ldots, V_n\} \cong K_t$. The assertions in these two steps clearly imply the first inequality of our theorem.

Note that the sets $V_1, V_2, \ldots, V_t$ depend only on the $T-W$ edges of the graph. Thus the edges joining the vertices of $W$ are chosen independently with probability $\frac{1}{t}$. Put $W_i = V_i - T$. We say that $(W_i, W_j), i \neq j$, is good if there is a $W_i - W_j$ edge. Since $W_i \subseteq W$ and $|W_i| = k$, clearly

$$P(\text{the pair } (W_i, W_j) \text{ is bad}) = 2^{-k^2}$$

and so the expected number of bad pairs is

$$\binom{t}{2} 2^{-k^2} = \frac{n^2}{\log_2 n} 2^{-\log_2 n - (\log_2 n)^t} = \frac{n}{\log_2 n} 2^{-\frac{1}{2} (\log_2 n)^t}.$$

At this stage we have several options. We may appeal either to the classical De Moivre–Laplace theorem (see [2; p. 134]) or to the even simpler Chebyshev inequality (see [2; p. 134]) or to the trivial inequality $P(|X| > |c|) \leq E(|X|)/|c|$ to deduce that almost every graph has few bad pairs. For example, the last inequality implies that the probability that a graph has more than

$$\frac{n}{\log_2 n} 2^{-\frac{1}{2} (\log_2 n)^t}$$

bad pairs is at most $2^{-\frac{1}{2} (\log_2 n)^t}$. In particular, since

$$t - \frac{n}{\log_2 n} 2^{-\frac{1}{2} (\log_2 n)^t} > s,$$

for almost every graph we can find sets $W_{n_1}, W_{n_2}, \ldots, W_{n_t}$ such that every pair $(W_{n_i}, W_{n_j})$ is good. Then we have $G / \{V_{n_1}, \ldots, V_{n_t}\} \cong K_t$ and since each $G[V_i]$ is connected, $\text{cc}(G) \geq s$. as claimed.

The proof of our theorem is complete.

With a little more effort the lower bound can be improved to $n((\log_2 n)^t + 1)^t$. Furthermore, the calculations can easily be carried over to the general case. If $0 < p < 1$ is fixed then almost every graph in $\mathcal{G}_n$ (the probability $p(\text{edge}) = p$) satisfies the inequality in the Theorem. with $\log_2 n$ replaced by $\log_b n$, where $b = 1/q$.

**References**


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