Topological Cliques of Random Graphs

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Given a graph $G$, denote by $tcl(G)$ the largest integer $r$ for which $G$ contains a
$TK'$, a topological complete $r$-graph. We show that for every $\varepsilon > 0$ almost every
graph $G$ of order $n$ satisfies

$$(2 - \varepsilon) n^{1/2} < tcl(G) < (2 + \varepsilon) n^{1/2}.$$ 

Throughout this paper we follow the notation and terminology of [1]. In
particular, $\chi(G)$ is the chromatic number of a graph $G$, $I(x)$ is the set of
neighbours of a vertex $x$ and $TK'$ is a topological complete $r$-graph, that is, a
graph homeomorphic to a complete $r$-graph $K'$. We also define the
topological clique number $tcl(G)$ of $G$ as $tcl(G) = \max\{r: G \supset TK'\}$.

A conjecture of Hajós, stating that $tcl(G) \geq \chi(G)$, had been open for over
25 years before Catlin [4] disproved it by exhibiting counterexamples for
$\chi(G) \geq 7$. Catlin's disproof of this conjecture prompted Erdős and Fajtlowicz
[5] to notice that almost every graph is a counterexample to the Hajós
conjecture. (For the basic properties of random graphs see [2, Chap.VIII].)
For it is well known (see [2] or [3] for a sharp result) that for $\varepsilon > 0$ almost
every (a.e.) graph $G$ of order $n$ satisfies

$$
\left(\frac{\log 2}{2} - \varepsilon\right) \frac{n}{\log n} < \chi(G) < (\log 2 + \varepsilon) \frac{n}{\log n}.
$$

On the other hand, it is shown in [5] that for some $c_1 > 0$

$tcl(G) < c_1 n^{1/2}$

for a.e. graph $G$ of order $n$. Thus

$$
\chi(G)/tcl(G) > c_2 n^{1/2}/\log n
$$

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for some positive constant $c_2$ and a.e. graph $G$ of order $n$. The aim of this paper is to show that $\text{tcl}(G)$ is about $2n^{1/2}$ for a.e. graph $G$ of order $n$. Consequently for every $\varepsilon > 0$ and a.e. $G$,

$$\left( \frac{\log 2}{4} - \varepsilon \right) \frac{n^{1/2}}{\log n} < \chi(G)/\text{tcl}(G) < \left( \frac{\log 2}{2} + \varepsilon \right) \frac{n^{1/2}}{\log n}.$$  

**Theorem.** Let $\varepsilon > 0$. Then a.e. graph $G$ of order $n$ satisfies

$$(2 - \varepsilon) n^{1/2} < \text{tcl}(G) < (2 + \varepsilon) n^{1/2}.$$  

Furthermore, a.e. $G$ is such that every set $W$ of $m = \lfloor (2 - \varepsilon) n^{1/2} \rfloor$ vertices $G$ is the set of branch vertices of a $TK^n$.

**Proof.** Stirling's formula has the following easy and well-known consequence. The probability that after $n$ tosses of an unbiased coin the difference between heads and tails is at least $\varepsilon n$ is not more than $(2/\varepsilon) e^{-\varepsilon n^{1/2}}$. Similarly, if heads occur with probability $p$, $0 < p < 1$, then the probability of having less than $(p - \varepsilon)n$ heads or more than $(p + \varepsilon)n$ heads is at most $e^{-cn}$, where $c > 0$ depends only on $p$ and $\varepsilon$. This allows us to deduce the following simple properties of almost all graphs.

(i) If $\varepsilon > 0$ is fixed and $m/\log n \to \infty$ then a.e. graph is such that every subgraph $H$ spanned by $m$ vertices satisfies

$$\left( \frac{1}{4} - \varepsilon \right) m^2 \leq e(H) \leq \left( \frac{1}{4} + \varepsilon \right) m^2.$$  

Indeed, the probability that a given $H$ fails to satisfy the inequalities is at most $e^{-\varepsilon m^2}$ provided $n$ is large enough. There are $\binom{n}{m}$ choices for $H$ and as $n \to \infty$,

$$\binom{n}{m} e^{-\varepsilon m^2} \leq n^m e^{-\varepsilon m^2} \to 0.$$  

(ii) Given $k \in \mathbb{N}$ and $\delta > 0$, a.e. graph $G$ is such that whenever $x_1, x_2, \ldots, x_{2k}$ are distinct vertices,

$$\left| \bigcup_{i=1}^{k} (I(x_{2i-1}) \cap I(x_{2i})) \right| \geq (1 - (\frac{1}{4})^k - \delta)n.$$  

Indeed, let $x_1, x_2, \ldots, x_{2k}$ be fixed and choose a vertex $x \in V(G) - \{x_1, x_2, \ldots, x_{2k}\}$. The probability that for a given $i$ the vertex $x$ belongs to $I(x_{2i-1}) \cap I(x_{2i})$ is $\frac{1}{4}$, so the probability that $x$ belongs to $\bigcup_{i=1}^{k} (I(x_{2i-1}) \cap I(x_{2i}))$ is $1 - (\frac{1}{4})^k$. Therefore the probability of (ii) failing is at most $e^{-cn}$ for
some constant $c > 0$. Since there are fewer than $n^{2k}$ choices for $x_1, x_2, \ldots, x_{2k}$, and

$$n^{2k} e^{-cn} \to 0 \quad \text{as} \quad n \to \infty,$$

the assertion follows.

The theorem is now easily proved

(a) Suppose $G$ contains a $TK^m$ whose set of branch vertices is $W$. If $x, y \in W$ are non-adjacent then the topological $x - y$ edge (which is $n$ $x - y$ path) contains at least one vertex not in $W$. As the topological edges are disjoint, in $G[W]$ at most $n - m$ edges are missing. We know from (i) that a.e. graph is such that from every subgraph spanned by $\lceil (2 + \epsilon) n^{1/2} \rceil$ vertices more than $n$ edges are missing. Hence almost no graph contains a $TK^m$ with $m = \lceil (2 + \epsilon) n^{1/2} \rceil$.

(b) Let $0 < \epsilon < \frac{1}{4}$, $m = \lceil (2 - \epsilon) n^{1/2} \rceil$ and choose $k_0 \in \mathbb{N}$ so that $(\frac{1}{2})^{k_0} \in \epsilon/2$. Put $\delta = \epsilon/2 - (\frac{1}{4})^{k_0}$. Almost every graph is such that from every subgraph spanned by $m$ vertices less than $(1 - (\epsilon/2))n$ edges are missing and (*) is satisfied for $k \leq k_0$.

Let $G$ be a graph with the properties above and let $W$ be any set of $m$ vertices of $G$. Let $F \subset W^{(2)}$ be the set of pairs of non-adjacent vertices of $W$. (Thus $F$ is the set of non-edges of $G[W]$.) By our choice of $G$ we have $|F| < (1 - \epsilon/2)n$. Define a bipartite graph with vertex classes $F$ and $Z = V(G) - W$ by joining $xy \in F$ to $Z \in Z$ if $z$ is a common neighbour of $x$ and $y$. Note that every edge of $B$ corresponds to a path of length 2 joining two vertices of $W$. In order to show that $W$ is the set of branch vertices of a $TK^m$, it suffices to show that $B$ has a matching from $F$ into $Z$, for then a $TK^m$ can be obtained by adding appropriate paths of length 2 to $G[W]$.

To complete the proof we have to check only that the condition of Hall's theorem [6] (see also [1, pp. 9 and 52]) is satisfied. If a set $F' \subset F$ contains a set $F''$ of $k_0$ independent non-edges, then by (*) the graph $B$ satisfies $|\Gamma(F')| \geq |\Gamma(F'')| \geq (1 - (\frac{1}{2})^{k_0} - \delta)n = (1 - \epsilon/2)n \geq |F| \geq |F'|$. On the other hand, if $F'$ does not contain $k_0$ independent non-edges then trivially

$$|F'| \leq 2k_0m$$

(for sharper estimates see [1, p. 58]). To estimate $\Gamma(F')$ all we need to note is that if $x_1, x_2 \in F'$ then by (*) applied with $k = 1$, for every sufficiently large $n$ we have

$$\Gamma(F') \geq |\Gamma(x_1) \cap \Gamma(x_2)| \geq (\frac{1}{4} - \delta)n \geq 2k_0m \geq |F'|.$$

The theorem can easily be carried over to the case when the edges of $G$ are chosen independently and with a fixed probability $p$, that is, if we
consider the probability space $\mathcal{G}(n, P(\text{edge}) = p)$ of [2, p. 123]. For every $\varepsilon > 0$ a.e. $G \in \mathcal{G}(n, P(\text{edge}) = p)$ satisfies
\[
\left( \frac{2}{1 - p} \right)^{1/2} - \varepsilon \left\{ n^{1/2} < \text{tcl}(G) < \left( \frac{2}{1 - p} \right)^{1/2} + \varepsilon \right\} n^{1/2}.
\]

In conclusion we note another related result. Denote by $TK^{m:s}$ a topological $K^m$ obtained from a $K^m$ by subdividing every edge into exactly $s$ edges. Define $k = k(s, n)$ to be the maximal integer satisfying
\[
k + \left( \frac{k}{2} \right)(s - 1) \leq n.
\]

Thus $k$ is the maximal integer for which $K^n$ contains a $TK^{k:s}$. A slightly more complicated version of the last part of the proof (the application of Hall's theorem) gives the following result.

Let $s \geq 2$ be fixed. Then for a.e. $G \in \mathcal{G}(n, P(\text{edge}) = p)$ we have
\[
\max \{m: G \supset TK^{m:s}\} = k(s, n).
\]

Furthermore, a.e. $G$ is such that every set of $k$ vertices is the set of branch vertices of some $TK^{k:s}$.

**REFERENCES**