Notation:

- $G$: a graph, with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$, and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$. 

$D(G)$: an orientation of $G$.

$D = (d_{ij})_{n \times m}$: vertex-edge incidence matrix, where $d_{ij}$ is defined as:

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\begin{cases} 
1 & \text{if } e_j \text{ is oriented away from } v_i \\
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A function $f : E \rightarrow A$ can be viewed as an $m$-dimensional vector

$$f = (f(e_1), f(e_2), \ldots, f(e_m))^T.$$
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$A$: is an abelian (additive) group with identity $0$, and with $|A| \geq 3$. 

Circular Flows of Graphs – p. 3/16
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- $A$ is an abelian (additive) group with identity $0$, and $|A| \geq 3$.

- $A^* = A \setminus \{0\}$: set of all non-zero elements of $A$. 
Nowhere-zero $A$-flows (or $A$-NZFs)

**Assumption:** For any graph $G$, we assume that a fixed orientation $D(G)$ of $G$ is given.

**Notation:** $\forall a \in A, 1 \cdot a = a, (-1) \cdot a = -a$ (additive inverse of $a$ in $A$), and $0 \cdot a = 0$ (additive identity of $A$)
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- For any $f \in F(G, \mathbb{A})$, the boundary of $f$ is $\partial f := Df$. That is, $\forall v_i \in V, \partial f(v_i) = Df(v_i)$, which is the $v_i$th component of the vector $Df$. 
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- A function $f \in F^*(G, A)$ is a nowhere-zero $A$-flow (or just an $A$-NZF) if $Df = 0$ (the all zero vector).
**Integer Flows**

- $\mathbb{Z}$: the abelian group of integers.

A function $f : F(G; \mathbb{Z})$ is a nowhere-zero $k$-flow (or just a $k$-NZF) if $Df = 0$, and if $0 < |f(e)| < k$.

Tutte: If $G$ has a $k$-NZF, then $G$ has a $(k + 1)$-NZF.

Tutte: A graph $G$ has an $A$-NZF if and only if $G$ has an $A$-NZF.
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If some orientation $D(G)$ has an $A$-NZF or a $k$-NZF, then for any orientation of $G$ also has the same property, and so having an $A$-MZF or a $k$-NZF is independent of the choice of the orientation.
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- If for an abelian group $A$, a connected graph $G$ has an $A$-NZF, then $G$ must be 2-edge-connected. (That is, $G$ does not have a cut edge).
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- If for an abelian group \( A \), a connected graph \( G \) has an \( A \)-NZF, then \( G \) must be 2-edge-connected. (That is, \( G \) does not have a cut edge).

- We shall only consider 2-edge-connected graphs \( G \) and define

\[
\Lambda(G) = \min\{ k : \ G \text{ has a } k - \text{NZF} \}. \]
Tutte’s Conjectures

- (5-flow) If $\kappa'(G) \geq 2$, then $\Delta(G) \leq 5$. 

(Jaeger’s weak 3-flow conjecture) There exists an integer $k > 0$ such that every $k$-edge-connected graph has a 3-NZF.
Tutte’s Conjectures

- (5-flow) If $\kappa'(G) \geq 2$, then $\Lambda(G) \leq 5$.

- (4-flow) If $\kappa'(G) \geq 2$, and if $G$ does not have subgraph contractible to $P_{10}$, the Petersen graph, then $\Lambda(G) \leq 4$. 
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- (3-flow) If $\kappa'(G) \geq 4$, then $\Lambda(G) \leq 3$.
- (Jaeger’s weak 3-flow conjecture) There exists an integer $k > 0$ such that every $k$-edge-connected graph has a 3-NZF.
Nowhere zero flows and colorings

Tutte: For a plane graph $G$, $G$ has a face $k$-coloring if and only if $G$ has a $k$-NZF.
Nowhere zero flows and colorings

- Tutte: For a plane graph $G$, $G$ has a face $k$-coloring if and only if $G$ has a $k$-NZF.
- These conjectures are theorems when restricted to planar graphs (need 4 Color Theorem for the 4-flow conjecture).
What do we know?

- Jaeger (1979): Every 2-edge-connected graph has a 8-NZF.
- Jaeger (1979): Every 4-edge-connected graph has a 4-NZF.
- Seymour (1981): Every 2-edge-connected graph has a 6-NZF.

The 5-flow conjecture and 3-flow conjecture have also been verified for projective planes and some other surfaces.
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- **Z. H. Chen, H. Y. Lai and HJL (2002):** Tutte’s flow conjectures are valid if and only if they are valid within line graphs.
Let $D = D(G)$ be an orientation of $G$. $\forall X \subset V(G)$, $\delta^+(X) = \text{set of edges going out from } X$, and $\delta^-(X) = \text{set of edges coming into } X$, and $\delta(X) = \delta^+(X) \cup \delta^-(X)$. Jaeger showed if

$$\phi_c(G) = \min_{D(G)} \max_{\emptyset \neq X \subset V(G)} \frac{|\delta(X)|}{|\delta^+(X)|},$$

then $\Lambda(G) = \lceil \phi_c(G) \rceil$. Therefore, $\phi_c(G)$, known as the circular flow number of $G$, is a refinement of $\Lambda(G)$, the flow number of $G$. 
Question: If edge connectivity is high? will $\phi_c(G)$ be less than 4?
Question: If edge connectivity is high? will $\phi_c(G')$ be less than 4?

Theorem: (A. Galluccio and L. A. Goddyn (2002), and R. Xu, C. Q. Zhang and HJL (2007)) If $\kappa'(G') \geq 6$, then $\phi_c(G') < 4$. 

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Theorem: (A. Galluccio and L. A. Goddyn (2002), and R. Xu, C. Q. Zhang and HJL (2007)) If $\kappa'(G) \geq 6$, then $\phi_c(G) < 4$.

Theorem: Jaeger (1979): Every 4-edge-connected graph has a 4-NZF.
Circular Flows of Graphs: Proof

- $G$ has a 4-NZF $\iff \exists g : E(G) \mapsto \{1, 2, 3\}$ such that $\partial g(v) = 0, \forall v \in V(G)$, under a given orientation $D$. 
Circular Flows of Graphs: Proof

- $G$ has a 4-NZF $\iff \exists g : E(G) \mapsto \{1, 2, 3\}$ such that $\partial g(v) = 0$, $\forall v \in V(G)$, under a given orientation $D$.

- Suppose that $\phi_c(G) = 4$. Then for some $X \subset V(G)$,

$$|\delta^+(X)| + |\delta^-(X)| = |\delta(X)| < 4|\delta^+(X)|.$$
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  This implies that $g^{-1}(2) \cap \delta(X) = \emptyset$.

- Idea: If we can find a positive 4-NZF $f$, such that $f^{-1}(2) \cap \delta(X) \neq \emptyset$, then we must have $\Lambda(G) < 4$. 

Circular Flows of Graphs – p. 13/16
Circular Flows of Graphs: Proof

By Nash-Williams and Tutte, $\kappa'(G) \geq 6$ implies $G$ has three edge-disjoint spanning trees $T_1, T_2$ and $T_3$. 
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- Let $C_1 = P_1 \cup P_2$, and $C_2 = G - E(P_1)$. Then both are even subgraphs (graphs with components being eulerian). Let $f_1 : C_1 \mapsto \{1\}$ and $f_2 : C_2 \mapsto \{2\}$, and $f = f_1 + f_2 : E(G) \mapsto \{1, 2, 3\}$ such that $T_3 \subseteq f^{-1}(2)$. 
New Conjectures

What we have proved: If $E(G)$ can be partitioned into $J_1 \cup J_2 \cup L_1$ such that each $J_i$ and $L_j$ is an $O(G)$-joint, and such that each $L_j$ is spanning and connected, then $\Lambda(G) < 4$.
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New Conjecture: If $E(G)$ can be partitioned into $J_1 \cup J_2 \ldots \cup L_t \cup L_1 \cup L_2 \ldots \cup L_k$ such that each $J_i$ and $L_j$ is an $O(G)$-joint, and such that each $L_j$ is spanning and connected, and if $t \geq 2$ and $k \geq 1$, then $\Lambda(G) \leq 2 + \frac{k+1}{k}$. 