On dense strongly $\mathbb{Z}_{2s+1}$-connected graphs

Aimei Yu a, Jianping Liu b, Miaomiao Han c, Hong-Jian Lai c

a Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
b College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350116, China
c Department of Mathematics, West Virginia University, Morgantown, WV 26506, United States

Abstract

Let $G$ be a graph and $s > 0$ be an integer. If, for any function $b : V(G) \rightarrow \mathbb{Z}_{2s+1}$ satisfying $\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2s+1}$, $G$ always has an orientation $D$ such that the net outdegree at every vertex $v$ is congruent to $b(v) \pmod{2s+1}$, then $G$ is strongly $\mathbb{Z}_{2s+1}$-connected. For a graph $G$, denote by $\alpha(G)$ the cardinality of a maximum independent set of $G$. In this paper, we prove that for any integers $s, t > 0$ and real numbers $a, b$ with $0 < a < 1$, there exist an integer $N(a, b, s)$ and a finite family $\mathcal{Y}(a, b, s, t)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph $G$ with order $n \geq N(a, b, s)$ and $\alpha(G) \leq t$, if $G$ satisfies one of the following conditions:

(i) for any edge $uv \in E(G)$, $\max\{d_G(u), d_G(v)\} \geq an + b$, or

(ii) for any $u, v \in V(G)$ with $\text{dist}_G(u, v) = 2$, $\max\{d_G(u), d_G(v)\} \geq an + b$,

then $G$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $G$ is not contractible to a member in the finite family $\mathcal{Y}(a, b, s, t)$.

1. Introduction

We consider finite graphs without loops, but multiple edges are allowed, and we follow [2] for undefined terms and notations. In particular, for a graph $G$, $\kappa'(G)$, $\delta(G)$ and $\alpha(G)$ denote the edge-connectivity, the minimum degree and the cardinality of a maximum independent set of $G$, respectively. Throughout this paper, $s > 0$ denotes an integer and $\mathbb{Z}$ denotes the set of all integers. For an $m \in \mathbb{Z}$, let $\mathbb{Z}_m$ be the set of integers modulo $m$, as well as the (additive) cyclic group on $m$ elements.

If $X \subseteq E(G)$, the contraction $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge in $X$ and then deleting the resulting loops. We define $G/\emptyset = G$. If $H$ is a subgraph of $G$, we write $G/H$ for $G/E(H)$. If $H$ is a connected subgraph of $G$ and $v_H$ is the vertex in $G/H$ onto which $H$ is contracted, then $H$ is the preimage of $v_H$ and is denoted by $\Pi_{v_H}(v_H)$.

Let $D$ denote an orientation of $G$. Following [2], for each $v \in V(G)$, we use $d_D^+(v)$ and $d_D^-(v)$ to denote the out-degree and the in-degree of $v$ under the orientation $D$, respectively. For an integer $m > 1$, if a graph $G$ has an orientation $D$ such that at every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{m}$, then we say that $G$ admits a mod-$m$ orientation. The set of all graphs which have mod-$m$ orientations is denoted by $M_m$. If $m = 2s$ is an even integer, then a connected graph $G$ is in $M_{2s}$ if and only if $G$ is Eulerian. Hence we are only interested in the case when $m = 2s + 1$ is an odd integer.

Let $A$ be an additive abelian group, and let $G$ be a graph with an orientation $D = D(G)$. For any vertex $v \in V(G)$, let $E_D^+(v)$ denote the set of all edges directed out from $v$, and let $E_D^-(v)$ denote the set of all edges directed into $v$. For a function
A function $f : V(G) \to A$ is a zero-sum function on $A$ if $\sum_{v \in V(G)} f(v) = 0$, where $0$ denotes the additive identity of $A$. The set of all zero-sum functions on $A$ of $G$ is denoted by $Z(G, A)$. Let $A'$ be a subset of $A$. We define $F(G, A') = \{ f : E(G) \to A' \}$. For any zero-sum function $b$ on $G$, a function $f \in F(G, A')$ satisfying $\partial f = b$ is referred to as an $(A', b)$-flow. When $b = 0$, an $(A - \{0\}, 0)$-flow is known as a nowhere-zero $A$-flow in the literature (see [9,10], among others). Following [10], if for any zero-sum function $b$ on $A$ of $G$ always has an $(A - \{0\}, b)$-flow, then $G$ is $A$-connected.

A graph $G$ is strongly $Z_m$-connected if, under a given orientation $D$, for any zero-sum function $b$ on $Z_m$ of $G$, there exists a function $f \in F(G, \{\pm 1\})$ such that $\partial f = b$. Again, for a given $b \in Z(G, Z_m)$ and an $f \in F(G, \{\pm 1\})$ with $\partial f = b$, one can keep the orientation of each edge with $f(e) = 1$ and reverse the orientation of each edge with $f(e) = -1$ to obtain a new orientation $D'$ of $G$ such that for any vertex $v \in V(G)$, $d^+_G(v) - d^-_G(v) = b(v) = \partial f(v)$. This orientation $D'$ will be referred to as a $(Z_m, b)$-orientation of $G$. Thus a graph $G$ is strongly $Z_m$-connected if and only if for any $b \in Z(G, Z_m)$, $G$ always has a $(Z_m, b)$-orientation. By definition, a graph $G$ is $Z_2$-connected if and only if $G$ is strongly $Z_2$-connected. But for an odd number $m \geq 5$, while every strongly $Z_m$-connected graph is $Z_m$-connected, not every $Z_m$-connected graph is strongly $Z_m$-connected. In Lemma 2.7 of [14](ii), it is known that $K_4$ is not strongly $Z_{2q+1}$-connected for any $q \geq 1$. However, as every edge of $K_4$ lies in a 3-cycle, it is known (see for example, [10], or Lemma 2.1 of [12]) that $K_4$ is $Z_m$-connected for any $m \geq 4$. It has been proved [14,15] that strongly $Z_{2k+1}$-connected graphs must be 2-edge-connected and are precisely the graphs $H$ such that for any graph $G$ containing $H$ as a subgraph, $G$ is in $M_{2k+1}$ if and only if $G/H$ is in $M_{2k+1}$. Therefore, following the notation of Catlin [3] and Catlin, Hobbs and Lai [4], the family of strongly $Z_{2k+1}$-connected graphs is denoted by $M_{2k+1}$.

Tutte and Jaeger proposed the following conjectures concerning mod $(2s + 1)$-orientations. A conjecture on strongly $Z_{2k+1}$-connected graphs has also been proposed recently.

**Conjecture 1.1.** Let $s > 0$ denote an integer.

(i) (Tutte [21]) Every 4-edge-connected graph has a mod 3-orientation.

(ii) (Jaeger [8,10]) Every 4s-edge-connected graph has a mod $(2s + 1)$-orientation.

(iii) (Jaeger [8,10]) Every 5-edge-connected graph is $Z_3$-connected.

(iv) [13,15] Every $(4s + 1)$-edge-connected graph is strongly $Z_{2s+1}$-connected.

**Theorem 1.1** (Thomassen [20]). Every 8-edge-connected graph is $Z_3$-connected.

**Theorem 1.2** (Lovász et al. [18], Wu [22]). Let $s > 0$ be an integer. Every 6s-edge-connected graph is strongly $Z_{2s+1}$-connected.

Barat and Thomassen presented the first degree condition to ensure a simple graph to be $Z_3$-connected. This was later improved by Fan and Zhou [5] and Luo et al. [19].

**Theorem 1.3** (Barat and Thomassen, Theorem 5.2 of [1]). There exists a positive integer $N$ such that every simple graph $G$ on $n \geq N$ vertices with $\delta(G) \geq n/2$ is $Z_3$-connected.

**Theorem 1.4** (Fan and Zhou [5]). Let $G$ be a simple graph on $n \geq 3$ vertices such that $d_c(u) + d_c(v) \geq n$, for every pair of nonadjacent vertices $u$ and $v$ in $G$. With six exceptional graphs, $G$ has a nowhere-zero 3-flow.

**Theorem 1.5** (Luo et al., Theorem 1.8 of [19]). Let $G$ be a simple graph on $n \geq 3$ vertices such that $d_c(u) + d_c(v) \geq n$, for every pair of nonadjacent vertices $u$ and $v$ in $G$. With 12 exceptional graphs, $G$ is $Z_3$-connected.

The results in Theorems 1.3–1.5 have the format that if a simple graph satisfies the Dirac condition or the Ore condition, then the graph has a nowhere-zero 3-flow or is $Z_3$-connected, with finitely many exceptional cases. This motivates the proofs of the following results.

**Theorem 1.6** (Li and Lai, Theorem 1.6 of [16]). Let $G$ be a simple graph on $n$ vertices. For any integer $s > 0$ and for any real numbers $a$ and $b$ with $0 < a < 1$, there exist an integer $N = N(a, b, s)$ and a finite family $\mathcal{F}(a, s)$ of non-$Z_{2k+1}$-connected graphs such that if $n \geq N$ and if $d_c(u) + d_c(v) \geq an + b$ for every pair of nonadjacent vertices $u$ and $v$ in $G$, then $G$ is strongly $Z_{2k+1}$-connected if and only if $G$ cannot be contracted to a member in $\mathcal{F}(a, s)$.

For positive integers $n$ and $s$ with $n \geq 2s$, let $K_{s,n-s}$ denote the simple graph obtained from the complete bipartite graph $K_{s,n-s}$ by adding one new edge joining two vertices of maximum degree.
Theorem 1.12. Let $G$ be a 2-edge-connected simple graph on $n$ vertices such that $dc(u) + dc(v) \geq n$, for every pair of adjacent vertices $u$ and $v$ in $G$. Then $G$ has a nowhere-zero 3-flow if and only if $G$ is not isomorphic to a $K_{3,n-3}$ or to one of the 5 other exceptional graphs.

Theorem 1.13 (Zhang et al., Theorem 1.3 of [24]). Let $G$ be a simple graph on $n \geq 3$ vertices such that $dc(u) + dc(v) \geq n$, for every pair of adjacent vertices $u$ and $v$ in $G$. Then $G$ is $\mathbb{Z}_3$-connected if and only if $G$ is not isomorphic to a member of $\{K_{2,n-2}, K_{2,n-2}, K_{3,n-3}, K_{3,n-3}^+\}$ or to one of the 15 other exceptional graphs.

Theorem 1.14 (Li et al., Theorem 1.4 of [17]). Let $G$ be a simple 2-edge-connected graph on $n \geq 3$ vertices. If for every $uv \not\in E(G)$, $\max(\{dc(u), dc(v)\}) \geq n/2$, then $G$ is $\mathbb{Z}_2$-connected if and only if $G$ is not contractible to one of 22 exceptional graphs.

Theorem 1.15. For any integer $s > 0$ and real numbers $a, b$ with $0 < a < 1$, there exist an integer $N = N(a, b, s)$ and a finite family $\mathcal{F}_0(a, s)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph $G$ with order $n \geq N$, if for any $uv \not\in E(G)$, $\max(\{dc(u), dc(v)\}) \geq an + b$, then $G$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $G$ cannot be contracted to a member in $\mathcal{F}_0(a, s)$.

For vertices $u, v$ in $G$, let $dist_c(u, v)$ denote the length of a shortest $(u, v)$-path in $G$.

Theorem 1.16 (Yan, Theorem 5.4 of [23]). Let $G$ be a 2-edge-connected simple graph on $n$ vertices. If for every pair $u, v \in V(G)$ with $dist_c(u, v) = 2$, $\max(\{dc(u), dc(v)\}) \geq n/2$, then $G$ is $\mathbb{Z}_3$-connected if and only if $G$ is not contractible to a member of a well defined graph family and $G$ is not isomorphic to one of 26 exceptional graphs.

Theorem 1.17. For any integers $s, t > 0$ and real numbers $a, b$ with $0 < a < 1$, there exist integers $N = \max(\lfloor \frac{12s-b-1}{a} \rfloor, \lfloor \frac{24s-2b-6}{a} \rfloor)$ and $B_1 = (12s - 1)(\lfloor \frac{1}{a} \rfloor + t - 1) + 1$ such that for any connected simple graph $G$ with order $n \geq N$ and $\alpha(G) \leq t$, if for every edge $uv \in E(G)$, $\max(\{dc(u), dc(v)\}) \geq an + b$, (1.1) then either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or $G$ is contractible to a member in $\mathcal{F}(B_1, s)$.

Theorem 1.18. For any integers $s, t > 0$ and real numbers $a, b$ with $0 < a < 1$, there exist integers $N = \max(\lfloor \frac{12s-b-1}{a} \rfloor, \lfloor \frac{24s-2b-6}{a} \rfloor)$ and $B_2 = (12s - 1)(\lfloor \frac{2}{a} \rfloor + 4st - 1) + 1$ such that for any connected simple graph $G$ with order $n \geq N$ and $\alpha(G) \leq t$, if for every pair $u, v \in V(G)$ with $dist_c(u, v) = 2$, $\max(\{dc(u), dc(v)\}) \geq an + b$, (1.2) then either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or $G$ is contractible to a member in $\mathcal{F}(B_2, s)$.

2. Some useful facts

Let $B, s > 0$ be integers. By definition, every graph in $\mathcal{F}(B, s)$ is not in $M^0_{2s+1}$. Lemma 2.1(i) indicates that any graph $G$ contractible to a member in $\mathcal{F}(B, s)$ cannot be in $M^0_{2s+1}$.

Lemma 2.1 ([13]). For any integer $s > 0$, each of the following holds.

(i) If $G$ is strongly $\mathbb{Z}_{2s+1}$-connected and $e$ is an edge of $G$, then $G/e$ is strongly $\mathbb{Z}_{2s+1}$-connected.

(ii) If $H$ is a subgraph of $G$ and both $H$ and $G/H$ are strongly $\mathbb{Z}_{2s+1}$-connected, then so is $G$.

Let $K^{(m)}_2$ denote the loopless graph with two vertices and $m$ parallel edges. Some examples of strongly $\mathbb{Z}_{2s+1}$-connected graphs have been found in [15].

Lemma 2.2 (Lemmas 2.2 and 2.7 of [14], and Liang [15]). Let $G$ be a graph and let $m, s > 0$ be integers. Each of the following holds:

(i) $K^{(m)}_2$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $m \geq 2s$;

(ii) $K^n$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $n = 1$ or $n \geq 4s + 1$. 
For any graph $G$, every vertex lies in a maximal strongly $Z_{2s+1}$-connected subgraph. Let $H_1, H_2, \ldots, H_e$ denote the collection of all maximal subgraphs that are in $M_{2s+1}^0$. Then $G' = G/(\bigcup_{j=1}^e E(H_j))$ is the $M_{2s+1}^0$-reduction of $G$. If $G$ is strongly $Z_{2s+1}$-connected, then its $M_{2s+1}^0$-reduction is $K_1$, a singleton. A graph which does not have any nontrivial subgraph in $M_{2s+1}^0$ is $M_{2s+1}^0$-reduced. Thus by definition, the $M_{2s+1}^0$-reduction of a graph is always $M_{2s+1}^0$-reduced. Even when $G$ is a simple graph, its $M_{2s+1}^0$-reduction may have multiple edges. Lemma 2.2 implies that there are only finitely many nontrivial $M_{2s+1}^0$-reduced graphs with a given order.

Define $\kappa'(G) = \max\{\kappa'(H) : H \subseteq G \text{ with } |E(H)| > 0\}$. The following is a corollary of Theorem 1.2.

**Lemma 2.3.** Let $G'$ be the $M_{2s+1}^0$-reduction of a connected graph $G$ such that $G' \neq K_1$. Then $\kappa'(G') \leq 6s - 1$.

For multigraphs with bounded values of $\kappa'$, the number of edges will also be bounded.

**Lemma 2.4** (Gu et al. [7]). Let $G$ be a graph with order $n$ and let $k > 0$ be an integer. If $\kappa'(G) \leq k$, then $|E(G)| \leq (n - 1)k$.

### 3. Proof of the main results

For notational convenience, throughout this section, we follow the notation in [3,4] to denote the family of strongly $Z_{2s+1}$-connected graphs by $M_{2s+1}^0$, and let $G$ be a connected simple graph on $n$ vertices. Let $G'$ be the $M_{2s+1}^0$-reduction of $G$ and $n' = |V(G')|$. Define

\[ X_c = \{v \in V(G') : d_{G'}(v) \leq c\}, \quad X'_c = \{v \in X_c : \text{Pl}_c(v) \neq K_1\}, \]
\[ Y = \{v \in V(G') : d_{G'}(v) \geq an + b\}, \quad W = \{v \in X'_c : \text{Pl}_c(v) \cap Y \neq \emptyset\}. \tag{3.3} \]

**Lemma 1.1.** If $G' \neq K_1$, each of the following holds.

(i) If $n \geq \max\{\lceil \frac{c-b}{a}\rceil, \lceil \frac{2c-2b-2}{a}\rceil\}$, then $|W| \leq \frac{2}{a}$.

(ii) If $c = 12s - 2$, then $n' - |X_c| \leq c|X_c| - 2(6s - 1)$.

**Proof.** (i) If $W = \emptyset$, $|W| \leq \frac{2}{a}$ holds immediately. If $W \neq \emptyset$, for any vertex $v \in W$, by the definition of $W$, there exists a vertex $u \in \text{Pl}_c(v) \cap Y$ such that

\[ |V(\text{Pl}_c(v))| - 1 + c \geq d_{G'}(u) \geq an + b. \]

This implies that $|V(\text{Pl}_c(v))| \geq an + b - c + 1$. Then

\[ |W|(an + b - c + 1) \leq \sum_{v \in W} |V(\text{Pl}_c(v))| \leq n. \tag{3.4} \]

As $n \geq \max\{\lceil \frac{c-b}{a}\rceil, \lceil \frac{2c-2b-2}{a}\rceil\}$, by (3.4), we have

\[ |W| \leq \frac{n}{an + b - c + 1} \leq \frac{2}{a}. \]

(ii) We first suppose that $V(G' - X_c) \neq \emptyset$ and $G' - X_c \neq K_1$. Since $G'$ is $M_{2s+1}^0$-reduced, $G' - X_c$ is also $M_{2s+1}^0$-reduced. By counting the edges in $G' - X_c$ and by Lemmas 2.3 and 2.4, we have

\[ (c + 1)(n' - |X_c|) - c|X_c| \leq 2|E(G' - X_c)| \leq 2(6s - 1)(n' - |X_c| - 1). \tag{3.5} \]

Set $c = 12s - 2$. By (3.5), we have

\[ n' - |X_c| = (c - 12s + 3)(n' - |X_c|) \leq c|X_c| - 2(6s - 1). \tag{3.6} \]

We now show that if $V(G' - X_c) = \emptyset$ or if $G' - X_c = K_1$, then $|X_c| \geq 2$, and so (3.6) holds as well. Since $G' \neq K_1$, $n' > 1$. If $V(G' - X_c) = \emptyset$, then $|X_c| = |V(G')| \geq 2$. Suppose that $G' - X_c = K_1$ but $|X_c| = 1$. Then $G' = K_2^{(n)}$. It contradicts the fact that $G'$ has a unique vertex with degree no more than $c$ and a unique vertex with degree no less than $c + 1$. Hence we must have $|X_c| \geq 2$ in this case. This completes the proof of Lemma 3.1.

Slightly stronger versions of Theorems 1.12 and 1.13 will be proved, which are presented as Theorems 3.1 and 3.2, respectively. By Lemma 2.1, if $G'$ is the $M_{2s+1}^0$-reduction of $G$, then $G \subseteq M_{2s+1}^0$, if and only if $G' \subseteq M_{2s+1}^0$. Therefore, the necessity of each of Theorems 3.1 and 3.2 is implied by Lemma 2.1. To prove the sufficiency of each of Theorems 3.1 and 3.2, we shall show that if $G$ is a graph satisfying the hypothesis and if $G$ is not in $M_{2s+1}^0$, then the $M_{2s+1}^0$-reduction $G'$ of $G$ must be in the specified finite family.

Recall that for integers $B, s > 0$, we define the family $\mathcal{F}(B, s)$ such that a graph $H$ is in $\mathcal{F}(B, s)$ if and only if $H \not\in M_{2s+1}^0$, $|V(H)| \leq B$ and $H$ is $M_{2s+1}^0$-reduced.
Theorem 3.1. For any integers \( s, t > 0 \) and real numbers \( a, b \) with \( 0 < a < 1 \), there exist an integer \( N = \max\left[ \left\lfloor \frac{12s-b-1}{a} \right\rfloor, \left\lfloor \frac{24s-2b-6}{a} \right\rfloor \right] \) and a finite family \( \mathcal{Y}_1(a, b, s, t) \) of non-strongly \( \mathcal{Z}_{2s+1} \)-connected graphs such that for any connected simple graph \( G \) with order \( n \geq N \) and \( \alpha(G) \leq t \), if (1.1) holds, then \( G \) is strongly \( \mathcal{Z}_{2s+1} \)-connected if and only if the \( M_0^0 \)-reduction of \( G \) is not a member in \( \mathcal{Y}_1(a, b, s, t) \).

Proof. Suppose that the real numbers \( a, b \) with \( 0 < a < 1 \) and the integers \( s, t \) are given. Set \( N = \max\left\{ \left\lceil \frac{12s-b-1}{a} \right \rceil, \left\lceil \frac{24s-2b-6}{a} \right \rceil \right\} \), and let \( \mathcal{Y}_1(a, b, s, t) \) denote the family of all connected simple graphs of order \( n \geq N \) and satisfying (1.1) with \( \alpha(G) \leq t \).

Define
\[
\mathcal{Y}_1(a, b, s, t) = \{ G' \mid G' \text{ is the } M_2^{0^0} \text{-reduction of a graph } G \in \mathcal{Y}_1(a, b, s, t) \} \setminus \{ K_1 \}.
\]

Let \( G \in \mathcal{Y}_1(a, b, s, t) \) be a graph, let \( G' \) be the \( M_2^{0^0} \)-reduction of \( G \) and \( n' = |V(G')| \). Since \( K_1 \not\in \mathcal{Y}_1(a, b, s, t) \), if \( G \in M_2^{0^0} \), then the \( M_2^{0^0} \)-reduction of \( G \) is not in \( \mathcal{Y}_1(a, b, s, t) \). Now we assume that \( G \not\in M_2^{0^0} \). By Lemma 2.1(ii), the \( M_2^{0^0} \)-reduction of \( G \) is in \( \mathcal{Y}_1(a, b, s, t) \). It remains to show that \( \mathcal{Y}_1(a, b, s, t) \) is a finite family.

Since \( G \not\in M_2^{0^0} \), we have \( n' > 1 \). We first show that there exists an integer \( B_1 = B_1(a, b, s, t) \) such that \( n' \leq B_1 \). Define \( X_1, X_2, Y, W \) as in (3.3) and set \( c = 12s - 2 \). Since \( n \geq \left\lceil \frac{12s-b-1}{a} \right \rceil \), we have \( c < an + b \). Hence, for any vertex \( v \in X_1 \cap Y, \) \( d_T(v) \leq c < an + b \). As \( G \) satisfies (1.1), \( X_1 \cap Y \) is an independent set of \( G \) and \( X_2 = W \). Since \( n \geq \max\{ \left\lceil \frac{c-b+1}{a} \right \rceil, \left\lceil \frac{2c-2b-2}{a} \right \rceil \} \), it follows by Lemma 3.1(i) and \( \alpha(G) \leq t \) that
\[
|X_1| = |X_1| + |X_2| - 2(6s-1) + (2s-1) + t.
\]

By Lemma 3.1(ii) and (3.7),
\[
n' = |X_1| + (n' - |X_1|) \\
\leq |X_1| + c|X_1| - 2(6s-1) \\
\leq (c + 1) \left( \left\lceil \frac{2}{a} \right \rceil + t \right) - 2(6s-1) \\
= (12s-1) \left( \left\lceil \frac{2}{a} \right \rceil + t - 1 \right) + 1.
\]

Let \( B_1 = (12s-1)(\left\lceil \frac{2}{a} \right \rceil + t - 1) + 1 \). Then \( n' \leq B_1 \).

Since \( G' \) is \( M_2^{0^0} \)-reduced and \( 1 < n' \leq B_1 \), it follows by the definition of \( \mathcal{F}(B, s) \) that \( G' \in \mathcal{F}(B_1, s) \), and so \( \mathcal{Y}_1(a, b, s, t) \subseteq \mathcal{F}(B_1, s) \). By Lemma 2.2, \( \mathcal{F}(B_1, s) \) is a finite family, and so \( \mathcal{Y}_1(a, b, s, t) \) is a finite family. This completes the proof of Theorem 3.1.

Theorem 3.2. For any integers \( s, t > 0 \) and real numbers \( a, b \) with \( 0 < a < 1 \), there exist an integer \( N = \max\left[ \left\lfloor \frac{12s-b-1}{a} \right \rceil, \left\lfloor \frac{24s-2b-6}{a} \right \rceil \right] \) and a finite family \( \mathcal{Y}_2(a, b, s, t) \) of non-strongly \( \mathcal{Z}_{2s+1} \)-connected graphs such that for any connected simple graph \( G \) with order \( n \geq N \) and \( \alpha(G) \leq t \), if (1.2) holds, then \( G \) is strongly \( \mathcal{Z}_{2s+1} \)-connected if and only if the \( M_2^{0^0} \)-reduction of \( G \) is not a member in \( \mathcal{Y}_2(a, b, s, t) \).

Proof. Suppose that the real numbers \( a, b \) with \( 0 < a < 1 \) and the integers \( s, t \) are given. Set \( N = \max\left\{ \left\lceil \frac{12s-b-1}{a} \right \rceil, \left\lceil \frac{24s-2b-6}{a} \right \rceil \right\} \), and let \( \mathcal{Y}_2(a, b, s, t) \) denote the family of all connected simple graphs of order \( n \geq N \) and satisfying (1.2) with \( \alpha(G) \leq t \).

Define
\[
\mathcal{Y}_2(a, b, s, t) = \{ G' \mid G' \text{ is the } M_2^{0^0} \text{-reduction of a graph } G \in \mathcal{Y}_2(a, b, s, t) \} \setminus \{ K_1 \}.
\]

Let \( G \in \mathcal{Y}_2(a, b, s, t) \) be a graph, let \( G' \) be the \( M_2^{0^0} \)-reduction of \( G \) and \( n' = |V(G')| \). Since \( K_1 \not\in \mathcal{Y}_2(a, b, s, t) \), if \( G \in M_2^{0^0} \), then the \( M_2^{0^0} \)-reduction of \( G \) is not in \( \mathcal{Y}_2(a, b, s, t) \). Now we assume that \( G \not\in M_2^{0^0} \). By Lemma 2.1(ii), the \( M_2^{0^0} \)-reduction of \( G \) is in \( \mathcal{Y}_2(a, b, s, t) \). It remains to show that \( \mathcal{Y}_2(a, b, s, t) \) is a finite family.

Since \( G \not\in M_2^{0^0} \), we have \( n' > 1 \). We first show that there exists an integer \( B_2 = B_2(a, s, t) \) such that \( n' \leq B_2 \). Define \( X_1, X_2, Y, W \) as in (3.3) and set \( c = 12s - 2 \). Then the following claim holds.

Claim A. \(|X_1 - X_2'| + |X_2' - W| \leq 4st|\).
Define \( T = (X_c - X'_c) \cup (\bigcup_{v \in X'_c - W} P_L(v)) \). Since \( n \geq \lceil \frac{12c - b - 1}{a} \rceil \), we have \( c < an + b \), and so for any vertex \( u \in X_c - X'_c \), 
\( d_G(u) \leq c < an + b \). Hence \( T \subseteq V(G) - \gamma \). As \( G \) satisfies (1.2), every connected component of \( G[T] \) (respectively, \( G[X_c - X'_c] \), \( \bigcup_{v \in X'_c - W} P_L(v) \)) is a complete graph. Furthermore, a connected component of \( G[X_c - X'_c] \) (respectively, \( \bigcup_{v \in X'_c - W} P_L(v) \)) is also a connected component of \( G[T] \). Assume that \( G[X_c - X'_c] \) has \( t_1 \) connected components and \( G[\bigcup_{v \in X'_c - W} P_L(v)] \) has \( t_2 \) connected components, i.e., \( |X'_c - W| = t_2 \). Then \( G[T] \) has \( t_1 + t_2 \) connected components and so \( t_1 + t_2 \leq \alpha(G) \leq t \). Since each component of \( G[X_c - X'_c] \) is not strongly \( Z_{2s+1} \)-connected, by Lemma 2.2(ii), its order is no more than 4s.

Hence
\[
|X_c - X'_c| + |X'_c - W| \leq t_1 \cdot 4s + t_2 \leq 4s(t_1 + t_2) \leq 4s \cdot \alpha(G) \leq 4st.
\]

This justifies Claim A.

Since \( n \geq \max\lceil \frac{12c - b - 1}{a} \rceil , \lceil \frac{2c - 2b - 2}{a} \rceil \), it follows by Claim A and Lemma 3.1(i) that
\[
|X_c| = |W| + |X_c - X'_c| + |X'_c - W| \leq \left\lfloor \frac{2}{a} \right\rfloor + 4st.
\] (3.8)

By Lemma 3.1(ii) and (3.8),
\[
n' = |X_c| + (n' - |X_c|) \\
\leq |X_c| + c |X_c| - 2(6s - 1) \\
\leq (c + 1) \left( \left\lfloor \frac{2}{a} \right\rfloor + 4st \right) - 2(6s - 1) \\
= (12s - 1) \left( \left\lfloor \frac{2}{a} \right\rfloor + 4st - 1 \right) + 1.
\]

Let \( B_2 = (12s - 1) \left( \left\lfloor \frac{2}{a} \right\rfloor + 4st - 1 \right) + 1 \). Then \( n' \leq B_2 \).

Since \( G' \) is \( M_{2s+1} \)-reduced and \( 1 < n' \leq B_2 \), it follows by the definition of \( F(B, s) \) that \( G' \in F(B_2, s) \), and so \( y_2(a, b, s, t) \subseteq F(B_2, s) \). By Lemma 2.2, \( F(B_2, s) \) is a finite family, and so \( y_2(a, b, s, t) \) is a finite family. This completes the proof of Theorem 3.2.

Acknowledgments

The research of Aimei Yu is supported by the National Natural Science Foundation of China (No. 11371193), the Beijing Higher Education Young Elite Teacher Project (No. YETP0573), Fundamental Research Funds for the Central Universities of China (No. 2015109) and the foundation of China Scholarship Council. The research of Jianping Liu is supported by the National Natural Science Foundation of China (Nos. 111101087; 11471077), the Foundation to the Educational Committee of Fujian (Nos. JA13025; JA13034), and by the fund “Overseas Training Program of Excellent Talents in Universities of Fujian in 2013”.

References