

# Nowhere zero 4-flow in regular matroids

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## Abstract

Jensen and Toft [10] conjectured that every 2-edge-connected graph without a  $K_5$ -minor has a nowhere zero 4-flow. Walton and Welsh [24] proved that if a coloopless regular matroid  $M$  does not have a minor in  $\{M(K_{3,3}), M^*(K_5)\}$ , then  $M$  admits a nowhere zero 4-flow. In this note, we prove that if a coloopless regular matroid  $M$  does not have a minor in  $\{M(K_5), M^*(K_5)\}$ , then  $M$  admits a nowhere zero 4-flow. Our result implies the Jensen and Toft conjecture.

## 1. Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [4] for graphs, and Oxley [12] or Welsh [25] for matroids.

Throughout this paper,  $\mathbf{Z}$ ,  $\mathbf{Z}^+$  and  $\mathbf{Z}_n$  denote the additive group of the integers, the set of all positive integers, and the cyclic group of order  $n$ , respectively. To be consistent with the matroid terminology, a nontrivial 2-regular connected graph will be called a **circuit**, and a disjoint union of circuits a **cycle**. Note that as we allow empty unions, the empty set is also a cycle (in both graphs and matroids). For matroids  $N_1, N_2, \dots, N_k$ , let  $EX(N_1, N_2, \dots, N_k)$  denote the collection of matroids such that a matroid  $M \in EX(N_1, N_2, \dots, N_k)$  if and only if  $M$  does not have a minor isomorphic to any one in  $\{N_1, N_2, \dots, N_k\}$ . The Fano matroid  $F_7$  is the vector matroid over  $\text{GF}(2)$  of the following matrix  $A$ :

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Flow was initially defined for graphs. For a discussion on flow and flow conjectures, see Jaeger [9] or Zhang [26]. The definition of flow has a natural extension to regular matroids.

Let  $M$  be a regular matroid and  $D_M$  be its incidence matrix of circuits against elements. An **orientation**  $(w(D_M), w(D_{M^*}))$  is an assignment of  $+$ ,  $-$  signs to the “1” entries of  $D_M$  and  $D_{M^*}$ , respectively, so that the resulting matrices  $w(D_M)$  and  $w(D_{M^*})$  satisfy

$$w(D_M)w(D_{M^*})^T = 0.$$

Let  $A$  be an abelian group. For an element  $a \in A$ , and for integers  $+1, -1, 0$ , we adopt the convention to write  $(+1) \cdot a = a$ ,  $(-1) \cdot a = -a$  and  $0 \cdot a = 0$ . Let  $F^*(M, A) = \{f : E(M) \mapsto A \setminus \{0\}\}$  denote the set of all functions from  $E(M)$  into  $A \setminus \{0\}$ . A map  $f \in F^*(M, A)$  can be viewed as an  $|E(M)|$ -dimensional column vector. For a regular matroid  $M$  with an orientation  $(w(D_M), w(D_{M^*}))$ , a map  $f \in F^*(M, A)$  satisfying

$$w(D_{M^*}) \cdot f = 0$$

is a **nowhere zero  $A$ -flow** ( $A$ -NZF for short) of  $M$ . When  $A = \mathbf{Z}$ , a  $\mathbf{Z}$ -NZF  $f$  of  $M$  is called a **nowhere zero  $k$ -flow** ( $k$ -NZF for short) of  $M$  if  $\forall e \in E(M)$ ,  $0 < |f(e)| < k$ .

For positive integers  $k$  and  $m$ , an  **$m$ -cycle  $k$ -cover** of a matroid  $M$  is a family of cycles  $C_1, C_2, \dots, C_m$  of  $M$  such that every element of  $E(M)$  lies in exactly  $k$  members of these  $C_i$ 's. It has been observed that a graph  $G$  admits a 4-NZF if and only if  $G$  has a 3-cycle 2-cover (for example, see Zhang [26]). We shall show later in this section that this is also true for regular matroids.

**Proposition 1.1** Let  $M$  be a regular matroid. Then  $M$  admits a 4-NZF if and only if  $M$  has a 3-cycle 2-cover.

Tutte proposed the famous 4-flow conjecture as follows.

**Conjecture 1.2** (Tutte [21] and [22], Matthews [11]) Let  $G$  be a 2-edge-connected graph. If  $G$  does not have a Peterson graph minor, then  $G$  admits a 4-NZF.

Jensen and Toft presented a weaker form of Conjecture 1.2 in 1995.

**Conjecture 1.3** (Jensen and Toft [10]) Every 2-edge-connected graph without a  $K_5$ -minor has a 4-NZF.

The main objective of this paper is to prove Conjecture 1.3 by proving a strong result in the matroid context as follows.

**Theorem 1.4** If  $M$  is a coloopless regular matroid such that  $M \in EX(M(K_5), M^*(K_5))$ , then  $M$  admits a 4-NZF.

**Corollary 1.5** Conjecture 1.3 holds affirmatively.

**Proof** Let  $G$  be a 2-edge-connected graph without a  $K_5$ -minor. Then  $M(G)$  is graphic and thus is regular. Since  $M^*(K_5)$  is not graphic, neither  $M(K_5)$  nor  $M^*(K_5)$  is a minor of  $M(G)$ . By Theorem 1.4,  $M(G)$  has a 4-NZF.  $\square$

The definition of flow has no natural extension to binary matroids whereas cycle cover is defined for general matroids. In view of Proposition 1.1 and the excluded-minor characterization of regular matroids, Theorem 1.4 is equivalent to saying that if a coloopless binary matroid  $M \in EX(F_7, F_7^*, M(K_5), M^*(K_5))$ , then  $M$  has a 3-cycle 2-cover. In Section 3 we will show that this result can be extended in the following form.

**Corollary 1.6** Let  $M$  be a coloopless binary matroid. If  $M \in EX(F_7^*, M(K_5), M^*(K_5))$ ,  $M$  has a 3-cycle 2-cover.

As the matroid  $F_7^*$  does not have a 3-cycle 2-cover (to be shown in Section 3), Corollary 1.6 does not hold if  $F_7^*$  is not excluded.

In the remainder of this section we will introduce the relevant definitions and briefly review the relevant results. In Section 2, we extract a decomposition theorem for regular matroids without  $M(K_5)$  or  $M^*(K_5)$  minors from the well known decomposition theorems of Seymour [16] and Wagner [23]. In Section 3, this theorem will be employed to prove Theorem 1.4 and Corollary 1.6.

**Theorem 1.7** (Tutte [20], Brylawski [6], Arrowsmith and Jaeger [3]) Let  $M$  be a regular matroid and  $A$  be an abelian group of order  $k$ . Then  $M$  has an  $A$ -NZF if and only if  $M$  has a  $k$ -NZF.

**Proof of Proposition 1.1:** Let  $M$  be a regular matroid. If  $M$  has a 3-cycle 2-cover  $\{C_1, C_2, C_3\}$ . Then  $E(M) = C_1 \cup C_2$ . Let  $f_i : C_i \mapsto \{1\} \in \mathbf{Z}_2$ , for  $i \in \{1, 2\}$ . Since  $C_i$  is a cycle,  $f = (f_1, f_2) \in F^*(M, \mathbf{Z}_2 \times \mathbf{Z}_2)$  is a  $(\mathbf{Z}_2 \times \mathbf{Z}_2)$ -NZF. By Theorem 1.7,  $M$  has a 4-NZF. By reversing this argument, we can also construct a 3-cycle 2-cover from a  $(\mathbf{Z}_2 \times \mathbf{Z}_2)$ -NZF of  $M$ .  $\square$

The Four-Color theorem can be stated in terms of nowhere zero flows as follows.

**Theorem 1.8** (Appel and Haken [1], Appel, Haken and Hoch [2], Robertson, Sanders, Seymour and Thomas [13]) Every 2-edge-connected planar graph admits a 4-NZF.

Applying the Four-Color theorem, and the duality between colorings and nowhere zero flows, a result by Walton and Welsh implies the following.

**Theorem 1.9** (Walton and Welsh [24]) If  $M$  is a coloopless regular matroid such that  $M \in EX(M(K_{3,3}), M^*(K_5))$ , then  $M$  admits a 4-NZF.

Recently Robertson et. al. prove Conjecture 1.2 for cubic graphs.

**Theorem 1.10** (Robertson, Sanders, Seymour and Thomas, [14]) Every 2-edge-connected cubic graph without a minor isomorphic to the Petersen graph admits a 4-NZF.

## 2. Decomposition of Regular Matroids in $EX(M(K_5), M^*(K_5))$

In this paper, we use  $\Delta$  to denote both a set operator and a matroid operator. Given two sets  $X$  and  $Y$ , the symmetric difference of  $X$  and  $Y$  is defined as

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Now suppose that  $M_1, M_2$  are binary matroids on  $E_1$  and  $E_2$ , respectively. We follow Seymour [16] and define a new binary matroid  $M_1 \Delta M_2$  to be the matroid with ground set equal to  $E_1 \Delta E_2$  and with its set of cycles equal to

$$\{C_1 \Delta C_2 \subseteq E_1 \Delta E_2 : C_i \text{ is a cycle of } M_i, i = 1, 2\}.$$

Three special cases of this operation are introduced by Seymour ([16] and [17]) as follows.

- (i) If  $E_1 \cap E_2 = \emptyset$  and  $|E_1|, |E_2| < |E_1 \Delta E_2|$ ,  $M_1 \Delta M_2$  is a **1-sum** of  $M_1$  and  $M_2$ .
- (ii) If  $|E_1 \cap E_2| = 1$  and  $E_1 \cap E_2 = \{z\}$ , say, and  $z$  is not a loop or coloop of  $M_1$  or  $M_2$ , and  $|E_1|, |E_2| < |E_1 \Delta E_2|$ ,  $M_1 \Delta M_2$  is a **2-sum** of  $M_1$  and  $M_2$ .
- (iii) If  $|E_1 \cap E_2| = 3$  and  $E_1 \cap E_2 = Z$ , say, and  $Z$  is a circuit of  $M_1$  and  $M_2$ , and  $Z$  includes no cocircuit of either  $M_1$  or  $M_2$ , and  $|E_1|, |E_2| < |E_1 \Delta E_2|$ ,  $M_1 \Delta M_2$  is a **3-sum** of  $M_1$  and  $M_2$ .

For  $i = 1, 2, 3$ , an  $i$ -sum of  $M_1, M_2$  is denoted as  $M_1 \oplus_i M_2$ . The 1-sum  $M_1 \oplus_1 M_2$  is also written as  $M_1 \oplus M_2$ . Let  $R_{10}$  denote the vector matroid of the following matrix over  $GF(2)$ :

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is known that  $R_{10}^*$  is isomorphic to  $R_{10}$ . Based on the notion of matroid sums, Seymour proved the following decomposition theorem for regular matroids.

**Theorem 2.1** (Seymour [16]) Let  $M$  be a regular matroid. One of the following must hold.

- (i)  $M$  is graphic.
- (ii)  $M$  is cographic.
- (iii)  $M \cong R_{10}$ .
- (iv) For some  $i \in \{1, 2, 3\}$ ,  $M = M_1 \oplus_i M_2$  is the  $i$ -sum of two matroids  $M_1$  and  $M_2$ , each of which is isomorphic to a proper minor of  $M$ .

**Theorem 2.2** (Seymour, Proposition (2.9) of [16]) Let  $M$  be a binary matroid. Then each of the following holds.

- (i) Let  $(X_1, X_2)$  be an exact 3-separation of a binary matroid  $M$  with  $|X_1|, |X_2| \geq 4$ , and let  $Z$  be a 3-element set that is disjoint from  $E(M)$ . Then there are binary matroids  $M_1$  and

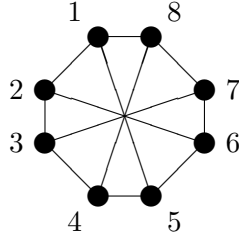
$M_2$  on  $X_1 \cup Z$  and  $X_2 \cup Z$ , respectively, such that  $M = M_1 \oplus_3 M_2$ .

(ii) If  $M$  is a 3-sum of  $M_1$  and  $M_2$ , then  $(E(M_1) \setminus E(M_2), E(M_2) \setminus E(M_1))$  is an exact 3-separation of  $M$ , and  $\min\{|E(M_1) \setminus E(M_2)|, |E(M_2) \setminus E(M_1)|\} > 3$ .

**Lemma 2.3** Let  $M$  be a 3-connected binary matroid such that  $M$  is a 3-sum of two matroids  $M_1$  and  $M_2$ . Then  $M^*$  is also a 3-sum.

**Proof** Define  $X_1 = E(M_1) \setminus E(M_2)$ ,  $X_2 = E(M_2) \setminus E(M_1)$ . By Theorem 2.2(ii),  $(X_1, X_2)$  is an exact 3-separation of  $M$  such that  $\min\{|X_1|, |X_2|\} \geq 4$ . Note that  $(X_1, X_2)$  is also an exact 3-separation of  $M^*$ , and that  $M^*$  is a 3-connected binary matroid. By Theorem 2.2(i),  $M^*$  must also be a 3-sum.  $\square$

If a matroid  $M$  is isomorphic to the cycle matroid of a planar graph, then  $M$  is called a **planar matroid**. Thus a matroid  $M$  is planar if and only if  $M^*$  is planar. Let  $H_8$  denote the graph depicted in the figure below.



**Figure 1:** The graph  $H_8$

Wagner's original statement of his decomposition theorem is in pure graph theory terms. A matroidal version is given as follows (see Seymour [16] and [17]).

**Theorem 2.4** (Wagner [23]) Let  $M$  be a graphic matroid that does not contain a minor isomorphic to  $M(K_5)$ . One of the following must hold.

(i)  $M$  is a planar matroid.

(ii)  $M \cong M(H_8)$ .

(iii)  $M \cong M(K_{3,3})$ .

(iv) For some  $i \in \{1, 2, 3\}$ ,  $M = M_1 \oplus_i M_2$  is the  $i$ -sum of two matroids  $M_1$  and  $M_2$ , such that both  $M_1$  and  $M_2$  are proper minors of  $M$ .

**Proposition 2.5** (Propositions 4.2.11, 8.3.1 and 12.4.16 of [12]) Each of the following holds:

(i) The matroid  $M$  is not 2-connected, if and only if for some proper non-empty subset  $T$  of  $E(M)$ ,  $M = (M|T) \oplus (M|(E \setminus T))$ . Note that  $M|T$  and  $M|(E \setminus T)$  are both proper minors of  $M$ .

(ii) The matroid  $M$  is 2-connected but not 3-connected if and only if  $M = M_1 \oplus_2 M_2$  for some matroids  $M_1$  and  $M_2$ , each of which is isomorphic to a proper minor of  $M$ .

(iii) If  $M$  is a 3-connected binary matroid and a 3-sum of  $M_1$  and  $M_2$ , then  $M_1$  and  $M_2$  are isomorphic to proper minors of  $M$ .

Let  $\mathcal{G}$  denote the family of matroids such that a matroid  $M \in \mathcal{G}$  if and only if  $M$  is a planar matroid or  $M$  is isomorphic to a member in the collection  $\{M(H_8), M^*(H_8), M(K_{3,3}), M^*(K_{3,3}), R_{10}\}$ . By definition, a matroid  $M \in \mathcal{G}$  if and only if  $M^* \in \mathcal{G}$ .

**Theorem 2.6** Let  $M$  be a regular matroid that does not have a minor isomorphic to  $M(K_5)$  or  $M^*(K_5)$ . Then one of the following must hold.

(i)  $M \in \mathcal{G}$ .

(ii) For some  $i \in \{1, 2, 3\}$ ,  $M = M_1 \oplus_i M_2$  is the  $i$ -sum of two matroids  $M_1$  and  $M_2$ , such that both  $M_1$  and  $M_2$  are proper minors of  $M$ .

**Proof** Let  $M$  be a regular matroid without a minor isomorphic to  $M(K_5)$  or  $M^*(K_5)$ , and such that  $M \notin \mathcal{G}$ . Since  $\mathcal{G}$  is closed under taking duals,  $M^* \notin \mathcal{G}$ . We shall show that  $M$  satisfies Theorem 2.6 (ii).

If  $M$  is not 2-connected, by Proposition 2.5 (i),  $M$  must be a 1-sum of two proper minors of  $M$ . If  $M$  is 2-connected but not 3-connected, by Proposition 2.5 (ii),  $M$  must be a 2-sum of two proper minors of  $M$ . In either case, Theorem 2.6 (ii) must hold.

Therefore we assume that  $M$  is 3-connected. Since  $M$  is regular, one of the conclusions of Theorem 2.1 must hold. Since  $R_{10} \in \mathcal{G}$ ,  $M$  cannot be  $R_{10}$ .

If  $M$  is graphic, then by Theorem 2.4 and since  $M$  is 3-connected, either  $M \in \mathcal{G}$ , or  $M$  is a 3-sum of two proper minors  $M_1$  and  $M_2$ , and so Theorem 2.6 (ii) must hold.

Now suppose that  $M$  is a 3-connected cographic matroid. Since  $M^* \notin \mathcal{G}$ , by Theorem 2.4,  $M^*$  must be a 3-sum of two proper minors. By Lemma 2.3,  $M$  must also be a 3-sum of two matroids  $M_1$  and  $M_2$ . By Proposition 2.5 (iii),  $M_1$  and  $M_2$  are isomorphic to proper minors of  $M$  and hence Theorem 2.6 (ii) must hold.

Now if  $M$  is neither graphic nor cographic, Theorem 2.6 (ii) follows from Theorem 2.1 (iv), and this completes the proof for Theorem 2.6.  $\square$

### 3. The Proofs of Theorem 1.4 and Corollary 1.6

In view of Proposition 1.1, we will prove Theorem 1.4 by showing that  $M$  has a 3-cycle 2-cover given the assumption of the theorem. We first establish some lemmas.

**Proposition 3.1** Each of the following holds.

- (i) Each of  $M(H_8)$ ,  $M^*(H_8)$ ,  $M(K_{3,3})$ ,  $M^*(K_{3,3})$ ,  $R_{10}$ ,  $F_7$  has a 3-cycle 2-cover.
- (ii)  $F_7^*$  cannot have a 3-cycle 2-cover.

These results follow from known facts about tangential 2-block. See for example the discussion on Tutte's tangential 2-block conjecture in [5]. The results can also be verified

directly in a straightforward way.

**Proposition 3.2** (Seymour [16], also see Oxley [12], Exercise 6 in Section 12.4) Suppose that  $M, M_1, M_2$  are binary matroids. If  $M = M_1 \triangle M_2$ , then  $M^* = M_1^* \triangle M_2^*$ .

**Lemma 3.3** Suppose that  $M, M_1, M_2$  are binary matroids and that each of  $M_1$  and  $M_2$  has a 3-cycle 2-cover. Then each of the following holds.

- (i) If  $M = M_1 \oplus M_2$  is a 1-sum of  $M_1$  and  $M_2$ , then  $M$  also has a 3-cycle 2-cover.
- (ii) If  $M = M_1 \oplus_2 M_2$  is a 2-sum of  $M_1$  and  $M_2$ , then  $M$  also has a 3-cycle 2-cover.

**Proof** (i) Suppose that  $M = M_1 \oplus M_2$ . For  $k = 1, 2$ , we assume that  $M_k$  has a 3-cycle 2-cover, denoted as  $C_{k,1}, C_{k,2}, C_{k,3}$ . It follows that  $\{C_{1,i} \cup C_{2,i} : i = 1, 2, 3\}$  is a 3-cycle 2-cover of  $M$ .

(ii) Now assume that  $M = M_1 \oplus_2 M_2$ . Denote  $E(M_1) \cap E(M_2) = \{e\}$ . For each  $k \in \{1, 2\}$ , assume that  $M_k$  has a 3-cycle 2-cover, denoted as  $C_{k,1}, C_{k,2}, C_{k,3}$ . Note that by the definition of a 2-cover,  $e$  appears exactly twice in each set of 3 cycles. Without loss of generality, we may assume that  $e \in C_{k,i}$ ,  $k, i = 1, 2$ . Now it is easy to verify that  $\{C_{1,i} \triangle C_{2,i} : i = 1, 2, 3\}$  is a 3-cycle 2-cover of  $M$ .  $\square$

Let  $\mathcal{F}_4$  denote the set of all matroids that have a 3-cycle 2-cover. Let  $\mathcal{F}$  denote the set of loopless and coloopless regular matroids which have no minors isomorphic to  $M(K_5)$  or  $M^*(K_5)$ . Note that  $\mathcal{F}$  is closed under taking duals and isomorphism, and that if  $M \in \mathcal{F}$  and if  $N$  is a loopless and coloopless minor of  $N$ , then  $N \in \mathcal{F}$  also. Thus for some  $i \in \{1, 2, 3\}$ , if  $M \in \mathcal{F}$  is an  $i$ -sum of  $M_1$  and  $M_2$ , each of which is a proper minor of  $M$ , then neither  $M_1$  nor  $M_2$  has any loop or coloop and therefore  $M_1$  and  $M_2$  are also in  $\mathcal{F}$ .

Let  $M$  be a matroid satisfying the hypothesis of Theorem 1.4 and let  $C$  be the union of all its loops. Then  $C$  is a cycle of  $M$  and  $M \setminus C$  is a matroid in  $\mathcal{F}$ , and  $M$  has a 3-cycle 2-cover if and only if  $M \setminus C$  has a 3-cycle 2-cover. Therefore, to prove Theorem 1.4, it suffices to prove that  $\mathcal{F} \subseteq \mathcal{F}_4$ .

**Proof of Theorem 1.4** Let  $M \in \mathcal{F}$ . If  $M \in \mathcal{G}$ , by Proposition 1.1, Theorem 1.8 and Proposition 3.1,  $M \in \mathcal{F}_4$ . We argue by induction on  $|E(M)|$  and assume that  $M \notin \mathcal{G}$  and that for every  $N \in \mathcal{F}$  with  $|N| < |M|$ ,  $N \in \mathcal{F}_4$ .

**Case 1:**  $M$  is not 3-connected.

By Proposition 2.5 (i) and (ii), for  $i \in \{1, 2\}$ ,  $M = M_1 \oplus_i M_2$  for some proper minors  $M_1, M_2$ . Hence by induction and by Lemma 3.3,  $M \in \mathcal{F}_4$ .

**Case 2:**  $M$  is 3-connected.

Since  $\mathcal{G}$  is closed under taking dual,  $M^* \notin \mathcal{G}$ . By Theorem 2.6,  $M^*$  must be a 1-, 2-, or 3-sum of two proper minors of  $M^*$ . Since  $M$ , and so  $M^*$  are 3-connected, it follows that  $M^* = M_1 \oplus_3 M_2$  is a 3-sum of some proper minor  $M_1$  and  $M_2$  of  $M^*$ . Denote  $E(M_1) \cap E(M_2) = \{e_1, e_2, e_3\} = Z$ . Then  $Z$  is a circuit of  $M_1$  and  $M_2$ .

By Proposition 3.2,  $M = M_1^* \triangle M_2^*$ . Note that  $M_1^*$  and  $M_2^*$  are proper minors of  $M$ , therefore  $M_1^*, M_2^* \in \mathcal{F}$ . For  $k = 1, 2$ , by induction,  $M_k^*$  has a 3-cycle 2-cover, denoted as

$C_{k,1}, C_{k,2}, C_{k,3}$ . Since  $Z$  is a cocircuit in binary matroids  $M_1^*$  and  $M_2^*$ ,  $|C_{k,i} \cap Z|$  must be even for any  $k \in \{1, 2\}$ ,  $i \in \{1, 2, 3\}$ . Therefore  $|C_{k,i} \cap Z| \in \{0, 2\}$ . As for each  $k \in \{1, 2\}$ ,  $\{C_{k,1}, C_{k,2}, C_{k,3}\}$  is a 2-cover of  $M_k$ , the following must hold.

$$\{C_{k,1} \cap Z, C_{k,2} \cap Z, C_{k,3} \cap Z\} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}\}.$$

Without loss of generality, we may assume that  $C_{k,i} \cap Z = Z - \{e_i\}$  for  $k \in \{1, 2\}$  and  $i \in \{1, 2, 3\}$ . Now it is easy to see that  $\{C_{1,i} \triangle C_{2,i} : i = 1, 2, 3\}$  is a 3-cycle 2-cover of  $M$ , and so  $M \in \mathcal{F}_4$ .

This proves Case 3 and thus completes the proof of Theorem 1.4.  $\square$

For binary matroids without  $F_7^*$  minor, Seymour has established the following decomposition theorem.

**Theorem 3.4** (Seymour [17]) Every binary matroid without  $F_7^*$  minor may be obtained by means of proper 1-sums or 2-sums from regular matroids and copies of  $F_7$ .

**Proof of Corollary 1.6** This follows from Proposition 3.1, Lemma 3.3, Theorem 3.4, and Theorem 1.4.  $\square$

## References

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