Open Problems in Group Connectivity and Group Colorings

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I. Notation and Terminology

(1.1) Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail*$(e) = u$ and *head*$(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{e \in E(D) \ : \ v = \text{ tail}(e)\}, \text{ and } E_D^+(v) = \{e \in E(D) \ : \ v = \text{ head}(e)\}.$$

The subscript D may be omitted when $D(G)$ is understood from the context.

Let A denote an (additive) Abelian group with identity 0, and let $F(G, A)$ denote the set of all functions from $E(G)$ to A. Given a function $f \in F(G, A)$, let $\partial f : V(G) \mapsto A$ be given by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Unless otherwise stated, we shall adopt the following convention: if $X \subseteq E(G)$ and if $f : X \mapsto A$ is a function, then we regard f as a function $f : E(G) \mapsto A$ such that $f(e) = 0$ for all $e \in E(G) - X$.

A function $b : V(G) \mapsto A$ is called an *A-valued zero sum function on G* if $\sum_{v \in V(G)} b(v) = 0$ in G. The set of all A-valued zero sum functions on G is denoted by $Z(G, A)$. Given $b \in Z(G, A)$, if G has an orientation G' and a function $f \in F^*(G, A)$ is an (A, b)-nowhere zero flow ((A, b)-NZF) if $\partial f = b$. A graph G is A-connected if for any b, G has an (A, b)-NZF. For an abelian group A, let $\langle A \rangle$ be the family of graphs that are A-connected. It is observed in [1] that the property $G \in \langle A \rangle$ is independent of the orientation of G, and that every graph in $\langle A \rangle$ is 2-edge-connected.

For a 2-edge-connected graph G, define

$$\Lambda_g(G) = \min\{k : G \in \langle A \rangle \text{ for any Abelian group } A \text{ with } |A| \geq k\}.$$

(1.2) For a function $f \in F(G, A)$, an **(A, f)-coloring** of G under the orientation D is a function c: $V(G) \mapsto A$ such that for any arc $e = uv \in E(G)$, $c(u) - c(v) \neq f(e)$; the graph

G is A**-colorable** under the orientation D if for any $f \in F(G, A)$, G has an (A, f)-coloring. For an Abelain group A, let $\langle A \rangle^*$ denote the set of all graphs that are A-colorable. (For any Abelian group A, whether G is A-colorable is independent of the choice of the orientation).

For a graph G, define

$$\chi_g(G) = \min\{m : \min\{k : G \in \langle A \rangle^* \text{ for any Abelian group } A \text{ with } |A| \geq k\}.$$

2. Problems in Group Connectivity

(2.1) **Conjecture:** (Jaeger et al, [1]) If G is a 3-edge-connected graph , then $\Lambda_g(G) \leq 5$.

(2.2) **Conjecture:** (Jaeger et al, [1]) If G is a 5-edge-connected graph , then $\Lambda_g(G) \leq 3$.

(2.3) **Problem** (Jaeger et al, [1]) Show that there exists an integer $k \geq 5$, such that every k-edge-connected graph G is in $\langle A \rangle$, for any A with $|A| \geq 3$.

(2.4) **Problem** Let $\{\kappa' \geq 3\}$ denote the family of all 3-edge-connected graphs. Is it true that for two Abelian groups A_1 and A_2, if $|A_1| = |A_2|$, then

$$\langle A_1 \rangle \cap \{\kappa' \geq 3\} = \langle A_2 \rangle \cap \{\kappa' \geq 3\}?$$

(2.5) **Problem** For two Abelian groups A_1 and A_2, if

$$\langle A_1 \rangle \cap \{\kappa' \geq 3\} = \langle A_2 \rangle \cap \{\kappa' \geq 3\},$$

does it imply that $|A_1| = |A_2|$?

(2.7) It is well known that every graph with a spanning eulerian subgraph (called a supereulerian graph) has a nowhere zero 4-flow. Is it true that if G is a 3-edge-connected supereulerian graph, then $\Lambda_g(G) \leq 4$?

3. Problems in Group Colorings

(3.1) **Conjecture** Let $k > 1$ be an integer. If G does not have a K_k-minor, then $\chi_g(G) \leq k$.

Remark: This may be the group coloring version of a Hadwiger conjecture. It holds when $k \leq 5$. (See [2] and [3]).

(3.2) **Conjecture** Let $\chi_l(G)$ denote the choice number of a graph G. Then $\chi_l(G) \leq \chi_g(G)$. (Evidence can be found in [2] and [3], especially [3]).

(3.3) **Problem** Let ß denote the family of all simple graphs. Is it true that for two abelain groups A_1 and A_2, if $|A_1| = |A_2|$, then

$$\langle A_1^* \rangle \cap \mathcal{S} = \langle A_2 \rangle^* \cap \mathcal{S}?$$

(3.4) **Problem** For two Abelian groups A_1 and A_2, if

$$\langle A_1 \rangle^* \cap \mathcal{S} = \langle A_2 \rangle^* \cap \mathcal{S},$$

does it imply that $|A_1| = |A_2|$?

References

[1] F. Jaeger, N. Linial, C. Payan, and M. Tarsi, Graph Connectivity of Graphs–A Non-homogeneous Analogue of Nowhere-Zero Flow Properties, J. Combin. Theory Series B 56, (1992), 165-182.

[2] H.-J. Lai and X. Zhang, Group colorability of graphs, Ars Combinatoria, 62 (2002), 299-317.

[3] H.-J. Lai and X. Zhang, Group chromatics number of graphs without K_5-minors, Graphs and Combinatorics, 18 (2002), 147-154.