On circular flows of graphs

Hong-Jian Lai[1], Rui Xu[2] and Cun-Quan Zhang[1*]

1 Department of Mathematics, West Virginia University, Morgantown WV 26506-6310

2 Department of Mathematics, University of West Georgia, Carrollton, GA 30118

Emails: hjlai@math.wvu.edu, xu@westga.edu, cqzhang@math.wvu.edu

Abstract

A sufficient condition for graphs with circular flow index less than 4 is found in this paper. In particular, we give a simple proof of a result obtained by Galluccio and Goddyn *(Combinatorica, 2002)*, and obtain a larger family of such graphs.

We refer readers to [1], [2] and [7] for the standard terminology and notations in this paper.

The following theorem was proved by Galluccio and Goddyn.

Theorem 1 (Galluccio and Goddyn [2]) *Let G be a 6-edge-connected graph. Then the circular flow index of G, $\phi_C(G) < 4$.*

Here, we give a simple proof of this theorem without using linear programming.

Proof. Since G is 6-edge-connected, by Tutte [5]-Theorem 1 or Nash-Williams [4]-Theorem 1, let T_1, T_2, T_3 be three edge disjoint spanning trees of G. Let P_i be a parity subgraph of T_i (for $i = 1, 2$ only). Now fixing some orientation of G. Since $P_1 \cup P_2$ and $G \setminus E(P_1)$ are even graphs, let f_1 be a nowhere-zero 2-flow with support $E(P_1) \cup E(P_2)$ and f_2 be a be a nowhere-zero 2-flow with support $E(G) \setminus E(P_1)$. Then $f = f_1 + 2f_2$ is a nowhere-zero 4-flow of G. Reorient the edges of G such that the resulting correspondent 4-flow $f^* > 0$. We will show this is the required orientation. First, this orientation is a strong orientation because it is easy to show that each edge is contained in a directed circuits. For each nonempty proper subset $X \subset V(G)$, the edge cut $\delta(X)$ contains at least one edge in T_3, hence having flow value 2. Therefore

$$3|\delta^+(X)| \geq \text{outflow of } X = \text{inflow of } X \geq |\delta^-(X)|,$$

*Partially supported by the National Security Agency under Grants MDA904-01-1-0022

with at least one strict inequality, as there is an edge in the cut of flow value 2. By the definition of the circular flow index (see [2]), $\phi_C(G) < 4$. ■

Similarly, we get the following results.

Theorem 2 *Let G be a graph. If G has a nontrivial parity subgraph decomposition such that at least one of its members is connected and spanning, then $\phi_c(G) < 4$.*

Theorem 3 *If a graph contains two edge-disjoint subgraphs P and H such that P is a parity subgraph and H is a connected, spanning collapsible subgraph of G, then $\phi_C(G) < 4$.*

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, 1976.

[2] A. Galluccio and L. A. Goddyn, The circular flow number of a 6-edge-connected graph is less than four, *Combinatorica*, 22 (2002), 455-459.

[3] L. A. Goddyn, M. Tarsi and C.-Q. Zhang, On (k, d)-colorings and fractional nowhere zero flows, *J. Graph Theory*, 28 (1998), 155-161.

[4] C. S. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.*, 36 (1961), 445-450.

[5] W. T. Tutte, On the problem of decompositing a graph into n connected factors, *J. London Math. Soc.*, 36 (1961), 221-230.

[6] C.-Q. Zhang, Circular flows of nearly eulerian graphs and vertex-splitting, *J. Graph Theory*, 40 (2002), 147–161.

[7] C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs,* Marcel Dekker Inc. New York, (1997) ISBN: 0-8247-9790-6.