

Compute partial derivatives with Chain Rule

Formulae: These are the most frequently used ones:

1. If $w = f(x, y)$ and $x = x(t)$ and $y = y(t)$ such that f, x, y are all differentiable. Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}. \quad (1)$$

2. If $w = f(x_1, x_2, \dots, x_m)$ and for each i , ($1 \leq i \leq m$), $x_i = x_i(t_1, t_2, \dots, t_n)$ such that f, x_1, \dots, x_m are all differentiable. Then

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_i}. \quad (2)$$

Example (1) : Given $w = \ln(u + v + z)$, with $u = \cos^2 t$, $v = \sin^2 t$ and $z = t^2$, find dw/dt both by using the chain rule and by expressing w explicitly as a function of t before differentiating.

Solution: First we apply Chain Rule (1):

$$\frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = \frac{1}{u + v + z} (-2 \cos t \sin t + 2 \sin t \cos t + 2t) = \frac{2t}{u + v + z}.$$

Next, we express w as a function of t by substituting u, v and z into w to get $w = \ln(\cos^2 t + \sin^2 t + t^2) = \ln(1 + t^2)$. Therefore, $dw/dt = 2t/(1 + t^2)$.

Example (2) : Given $w = yz + zx + xy$, $x = s^2 - t^2$, $y = s^2 + t^2$ and $z = s^2 t^2$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

Solution: This is a partial derivative problem and so we apply Chain Rule (2).

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} = (z + y)(2s) + (z + x)2s + (x + y)2st^2 \\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} = -(z + y)(2t) + (z + x)2t + (x + y)2s^2 t \end{aligned}$$

Example (3) : Given $p = f(x, y, z)$, $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, write the chain rule formulas giving the partial derivatives of the dependent variable p with respect to each independent variable.

Solution: This is a partial derivative problem and so we apply Chain Rule (2).

$$\begin{aligned} \frac{\partial p}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} \\ \frac{\partial p}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} \end{aligned}$$

Example (4) : Given $x^3 + y^3 + z^3 = xyz$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ as functions of x, y and z .

Solution: As z is an implicit function of x and y , implicit differentiation must be used. Just view $z = z(x, y)$ everywhere z occurs when we differentiate both sides of the equation.

(Step 1) View $z = z(x, y)$ and differentiate both sides of the equation with respect to x to get

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} = yz + xy \frac{\partial z}{\partial x}.$$

(Step 2) Solve for $\frac{\partial z}{\partial x}$ in the resulting equation above.

$$z_x = \frac{\partial z}{\partial x} = \frac{yz - 3x^2}{3z^2 - xy}.$$

Do the same for y .

(Step 1) View $z = z(x, y)$ and differentiate both sides of the equation with respect to y to get

$$3y^2 + 3z^2 \frac{\partial z}{\partial y} = xz + xy \frac{\partial z}{\partial y}.$$

(Step 2) Solve for $\frac{\partial z}{\partial y}$ in the resulting equation above.

$$z_y = \frac{\partial z}{\partial y} = \frac{xz - 3y^2}{3z^2 - xy}.$$

Example (5) : Given $x^3 + y^3 + z^3 = xyz$ as the equation of a surface, find an equation of the plane tangent to this surface at the point $P(1, -1, -1)$.

Solution 1: View $z = z(x, y)$, and so the vector $\mathbf{n} = (z_x, z_y, -1)$ at P will be a normal vector of the tangent plane. From Example (4) above, we have, at $P(1, -1, -1)$,

$$\begin{aligned} z_x &= \left. \frac{yz - 3x^2}{3z^2 - xy} \right|_{(x,y,z)=(1,-1,-1)} = \frac{(-1)(-1) - 3}{3(-1)^2 - 1(-1)} = \frac{-2}{4} = -\frac{1}{2}, \\ z_y &= \left. \frac{xz - 3y^2}{3z^2 - xy} \right|_{(x,y,z)=(1,-1,-1)} = \frac{1(-1) - 3}{3(-1)^2 - 1(-1)} = \frac{-4}{4} = -1. \end{aligned}$$

Hence the equation of the tangent plane is

$$-\frac{1}{2}(x - 1) - (y + 1) - (z + 1) = 0, \text{ or } (x - 1) + 2(y + 1) + 2(z + 1) = 0.$$

Solution 2: One can also set $F(x, y, z) = x^3 + y^3 + z^3 - xyz$ and view the equation of the surface is $F(x, y, z) = 0$. In this case, the vector $\mathbf{u} = (F_x, F_y, F_z)$ at $P(1, -1, -1)$ can be a normal vector of the tangent plane. We compute the partial derivatives:

$$F_x = \frac{\partial F}{\partial x} = 3x^2 - yz, \quad F_y = \frac{\partial F}{\partial y} = 3y^2 - xz, \quad F_z = \frac{\partial F}{\partial z} = 3z^2 - xy.$$

Therefore, $\mathbf{u} = (2, 4, 4)$ and so an equation of the tangent plane is

$$2(x - 1) + 4(y + 1) + 4(z + 1) = 0, \text{ or } (x - 1) + 2(y + 1) + 2(z + 1) = 0.$$

Example (6) : Suppose that $w = f(u)$ and that $u = x + y$. Show that $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}$.

Solution: Notice that $\frac{\partial u}{\partial x} = 1 = \frac{\partial u}{\partial y}$. By Chain Rule (2),

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y}.$$

Example (7) : Suppose that $w = f(x, y)$ and that $x = r \cos \theta$ and $y = r \sin \theta$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2}.$$

Solution: Apply Chain Rule (2) to compute the first order of partial derivatives

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} = w_x \cos \theta + w_y \sin \theta \\ \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} = -w_x r \sin \theta + w_y r \cos \theta. \end{aligned}$$

Then, we apply Chain Rule (2) again to compute the second order of partial derivatives (making use of $w_{xy} = w_{yx}$),

$$\begin{aligned} \frac{\partial^2 w}{\partial r^2} &= w_{xx} \cos^2 \theta + w_{xy} \cos \theta \sin \theta + w_{yx} \sin \theta \cos \theta + w_{yy} \sin^2 \theta \\ &= w_{xx} \cos^2 \theta + (w_{xy} + w_{yx}) \cos \theta \sin \theta + w_{yy} \sin^2 \theta, \end{aligned} \quad (3)$$

and

$$\begin{aligned} \frac{\partial^2 w}{\partial \theta^2} &= w_{xx} r^2 \sin^2 \theta - w_{xy} r^2 \sin \theta \cos \theta - w_x r \cos \theta + w_{yy} r^2 \cos^2 \theta + w_{yx} r^2 \cos \theta (-\sin \theta) - w_y r \sin \theta \\ &= w_{xx} r^2 \sin^2 \theta + w_{yy} r^2 \cos^2 \theta - (w_{xy} + w_{yx}) r^2 \sin \theta \cos \theta - w_x r \cos \theta - w_y r \sin \theta. \end{aligned}$$

It follows that

$$\frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = w_{xx} \sin^2 \theta + w_{yy} \cos^2 \theta - (w_{xy} + w_{yx}) \sin \theta \cos \theta - \frac{w_x \cos \theta + w_y \sin \theta}{r} \quad (4)$$

$$\frac{1}{r} \frac{\partial w}{\partial r} = \frac{w_x \cos \theta + w_y \sin \theta}{r}. \quad (5)$$

Now add equations (3), (4) and (5) side by side and apply $\sin^2 \theta + \cos^2 \theta = 1$ to get the conclusion.