

MATH 124 COLLEGE ALGEBRA LAB 8 RATIONAL FUNCTIONS

OBJECTIVES

- To explore the end behavior of rational graphs.
- To explore the relationship between the degrees of the numerator and the denominator of a rational function and its asymptotic behavior.
- To explore the intermediate behavior of rational graphs
- To find the complete graph for rational functions
- To find a rational function model.

Complete Graph of a Rational Function (Chapter 3: Sections 3-5 and 3-6)

In Lab 7 we explored the complete graph of a polynomial function. This included determining the following:


1. The intervals where the graph is **increasing** or **decreasing**. A graph is increasing if as you move from left to right on the x-axis, the associated y-values increase. A graph is decreasing if as you move from left to right on the x-axis, the associated y-values decrease.
2. The **number of turns** in the graph. A turn is when the direction of the graph changes from increasing to decreasing, or decreasing to increasing. The point at which a turn occurs is either a **maximum (“peak”)** or **minimum (“valley”)** value of the function.
3. The location of any **x-intercepts** and the **y-intercept**. Turns and intercepts of a graph are part of what is called the **intermediate (local) behavior** of the graph.
4. The **end behavior** of the graph. As opposed to the intermediate behavior, end behavior is what happens to the graph of the function at the extreme left and right ends of the x-axis.



We will now repeat this exploration for rational functions. Unlike polynomial functions, rational functions can have values where the function fails to exist. For example,

$$r(x) = \frac{1}{x}$$

fails to exist when $x=0$. So how does $r(x)$ behave as $x \rightarrow 0$?

Exercise 1: Enter the rational function $y = \frac{1}{x}$ in *Derive 5*. Plot the rational function by selecting

the 2D Window Plot icon  to move to the graphing window and the Plot Expression icon

. The Trace Plots icon  can be used to explore the behavior of the function as x approaches the undefined point from the left or right. Select the Trace Plot command and click on the right half of the graph. Now use the left and right arrow keys on your keyboard to trace the function. Trace the function by letting x approach $x = 0$ from the right. Examine the values of y as x approaches 0 from the right (*Derive 5* provides the values of x and y in the lower left part of the screen).

A. As $x \rightarrow 0$ from the right what does $r(x)$ approach?

B. As $x \rightarrow 0$ from the left what does $r(x)$ approach?

The vertical line through the point where a rational function does not exist is called a **vertical asymptote**. An asymptote is a limiting function for a graph; that is, the function approaches the asymptote as x approaches some real number value. For the basic rational function $y = 1/x$ the vertical asymptote is the line $x = 0$.

Exercise 2: Given any rational function we can find potential vertical asymptotes by setting the denominator equal to zero and solving. Use this method and the *Derive 5 Solve Expression* command to determine the vertical asymptotes for the rational function

$$y = \frac{x^2 - 3x}{x^2 - 5x - 9}$$

Plot the rational function and its vertical asymptotes. Notice how the vertical asymptotes act as limiting lines for the function; that is, the function approaches the vertical asymptotes but never crosses them.

Rational functions also differ from polynomial functions with respect to end behavior. Polynomial functions can have only one of four different end behaviors. Rational functions have asymptotes that determine end behavior. **End behavior asymptotes** are how the function behaves as $x \rightarrow -\infty$ or $x \rightarrow +\infty$.

Exercise 3: Examine the graph of the basic rational function $r(x) = \frac{1}{x}$.

A. Determine what the end behavior of the function is by using the Trace Plots command to determine what y approaches as x approaches positive infinity. As x approaches negative infinity.

B. Write an equation for the end behavior asymptote.

We will explore the behavior of rational function graphs with respect to asymptotes, both asymptotes that are part of the function's intermediate behavior (i.e.: vertical asymptotes) and asymptotes that are part of the function's end behavior (i.e.: horizontal asymptotes, slant asymptotes). Specifically, we explore the effect of degrees of the numerator and denominator of the rational function upon the type of asymptote.

Rational Functions where the Degree of Numerator < Degree of Denominator

Begin by examining rational functions where the degree of the numerator is less than the degree of the denominator. You can make up your own examples or use the one provided in the exercise.

Exercise 4: Make up a rational function where the degree of the numerator < degree of the denominator, such as:

$$r(x) = \frac{3x^2 + 6x - 7}{4x^5 - 9x + 9}$$

- A. Use *Derive 5* to plot the graph of the function (Do not use the **Option Autoscale New Plots**).
- B. Use the **Solve Expression Numeric Real** to determine the equation(s) of the vertical asymptotes of the function. Plot the vertical asymptotes.
- C. Identify the intermediate behavior around these asymptotes. As x approaches the vertical asymptote from the left what does $r(x)$ approach? As x approaches the vertical asymptote from the right what does $r(x)$ approach?
- D. Now examine the end behavior of the function $r(x)$. To do this ignore all but the highest degree term in the numerator and the denominator, we will call this the **end behavior model** for the rational function. The end behavior model for $r(x)$ is

$$y = \frac{3x^2}{4x^5} \text{ which reduces to } y = \frac{3}{4x^3}$$

Plot this model with the original function and examine what happens as $x \rightarrow -\infty$ and $x \rightarrow +\infty$. Does this model have the same end behavior as the original function?

- E. To find the exact **end behavior asymptote** we divide the denominator into the numerator and examine the result. As x approaches infinity any rational expression where the degree of the numerator is less than the degree of the denominator will approach 0 (Can you see why?), so the end behavior asymptote is the quotient (any constant expression or any expression where the degree of the numerator is greater than the degree of the denominator). Use the *Derive 5 Simplify Expand Command* to perform the division by expanding for variable x over the rationals. Determine the actual end behavior asymptote. What is the **end behavior asymptote** for any rational function where the degree of the numerator is less than the degree of the denominator?

Rational Functions where the Degree of Numerator = Degree of Denominator

How does a rational function with degree of numerator = degree of denominator differ from a rational function where the degree of numerator < degree of denominator? Let's explore.

Exercise 5: Repeat Exercise 4 for a rational function where the degree of the numerator = degree of the denominator. You can make up or own or use the following example:

$$r(x) = \frac{5x^3 - 8x^2 + 9}{-2x^3 + 2x - 8}$$

- A. Use *Derive 5* to plot the graph of the function (Do not use the **Option Autoscale New Plots**).
- B. Use the **Solve Expression Numeric Real** to determine the equation(s) of the vertical asymptotes of the function. Plot the vertical asymptotes.
- C. Identify the intermediate behavior around these asymptotes. As x approaches the vertical asymptote from the left what does $r(x)$ approach? As x approaches the vertical asymptote from the right what does $r(x)$ approach?
- D. Determine the **end behavior model** for the rational function. Plot this model with the original function and examine what happens as $x \rightarrow -\infty$ and $x \rightarrow +\infty$. Does this model have

the same end behavior as the original function?

E. Find the exact **end behavior asymptote** using the *Derive 5 Simplify Expand Command* to perform the division by expanding for variable x over the rationals. What is the **end behavior asymptote** for any rational function where the degree of the numerator equals the degree of the denominator?

Rational Function where Degree of Numerator > Degree of Denominator

How do rational functions where the degree of the numerator is greater than the degree of the denominator differ from the above cases? We will look at two examples, one where the degree of the numerator is only one greater than the denominator and one where the degree of the numerator is more than one greater than the degree of the denominator (see Data Sheet).

Exercise 6: Repeat Exercise 4 for a rational function where the degree of the numerator is one more than the degree of the denominator, such as:

$$r(x) = \frac{6x^4 - 7x^2 + 13}{4x^3 + 6x^2 - x - 1}$$

A. Use *Derive 5* to plot the graph of the function (Do not use the **Option Autoscale New Plots**).

B. Use the **Solve Expression Numeric Real** to determine the equation(s) of the vertical asymptotes of the function. Plot the vertical asymptotes.

C. Identify the intermediate behavior around these asymptotes. As x approaches the vertical asymptote from the left what does $r(x)$ approach? As x approaches the vertical asymptote from the right what does $r(x)$ approach?

D. Determine the **end behavior model** for the rational function. Plot this model with the original function and examine what happens as $x \rightarrow -\infty$ and $x \rightarrow +\infty$. Does this model have the same end behavior as the original function?

E. Find the exact **end behavior asymptote** using the *Derive 5 Simplify Expand Command* to perform the division by expanding for variable x over the rationals. What is the **end behavior asymptote** for any rational function where the degree of the numerator is greater than the degree of the denominator?