1. The limit \( \lim_{x \to 2} (x^2 - 2)(x^2 + x) \) is equal to

(A) 2  (B) 6  (C) 12

(D) 16  (E) 0  (F) None of These.

**Solution:** As \( \lim_{x \to 2} (x^2 - 2)(x^2 + x) = (2^2 - 2)(2^2 + 2) = 12 \), the answer is (C).

2. Given that \( \lim_{x \to a} f(x) = -2 \), \( \lim_{x \to a} g(x) = 5 \), and \( \lim_{x \to a} h(x) = 3 \),

the limit \( \lim_{x \to a} \frac{2f(x) + g(x)}{2h(x) - g(x)} \) is equal to

(A) 1  (B) 0  (C) \( \frac{1}{6} \)

(D) \( \frac{2}{3} \)  (E) DNE  (F) None of These.

**Solution:** As \( \lim_{x \to a} \frac{2f(x) + g(x)}{2h(x) - g(x)} = \frac{2\lim_{x \to a} f(x) + \lim_{x \to a} g(x)}{2\lim_{x \to a} h(x) - \lim_{x \to a} g(x)} = \frac{2(-2) + 5}{2 \cdot 3 - 5} = 1 \), the answer is (A).

3. The limit \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \) is equal to

(A) DNE  (B) 0  (C) 2  (D) 4

(E) \(-2\)  (F) \(-4\)  (G) 1  (H) None of these.

**Solution:** As \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4 \), the answer is (D).

4. Given the function

\[
f(x) = \begin{cases} 
2x + 1 & \text{if } x < 0 \\
\frac{x^2}{2} & \text{if } x > 0 
\end{cases}
\]

the value of \( f(0) \) is
(A) 1  (B) either $2x + 1$ or $x^2$  (C) not defined  (D) 0

(E) $x^2$  (F) either 1 or 0  (G) None of these  (H) $2x + 1$.

**Solution:** Since the function $f(x)$ only gives the value for $x < 0$ or $x > 0$, $f(x)$ is undefined at $x = 0$. The answer is (C).

5. The limit $\lim_{x \to 3^-} \frac{|x - 3|}{x - 3}$ is equal to

(A) 0  (B) $-\infty$  (C) DNE

(D) 1  (E) $-1$  (F) None of These.

**Solution:** Since $x$ approaches 3 from the left side, $x < 3$. Thus $|x - 3| = -(x - 3)$ when $x < 3$, and so $\lim_{x \to 3^-} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^-} \frac{- (x - 3)}{x - 3} = \lim_{x \to 3^-} (-1) = -1$. The answer is (E).

6. The limit $\lim_{x \to 0} \frac{x}{\cos x}$ is equal to

(A) 1  (B) $-1$  (C) 0

(D) $\pi$  (E) $-\pi$  (F) None of These.

**Solution:** $\lim_{x \to 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$. The answer is (C).

7. The limit $\lim_{x \to \pi} \frac{\sin(x - \pi)}{x - \pi}$ is equal to

(A) 1  (B) $-1$  (C) 0

(D) $-\infty$  (E) DNE  (F) None of These.

**Solution:** Let $y = x - \pi$. As $x \to \pi$, $y \to 0$. So $\lim_{x \to \pi} \frac{\sin(x - \pi)}{x - \pi} = \lim_{y \to 0} \frac{\sin y}{y} = 1$. The answer is (A).
8. Given the function
\[ f(x) = \begin{cases} 
2x + 1 & \text{if } x < 1 \\
4x^2 - 1 & \text{if } x > 1 
\end{cases} 
\]
which of the following statements is correct?
(A) \( f(x) \) is continuous at \( x = 1 \).
(B) The discontinuity of \( f(x) \) at \( x = 1 \) can be removed by defining \( f(1) = 2 \).
(C) \( f(x) \) has a non removable discontinuity at \( x = 1 \).
(D) The discontinuity of \( f(x) \) at \( x = 1 \) can be removed by defining \( f(1) = 3 \).
(E) None of above is correct.

Solution: Since
\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 1) = 2 \cdot 1 + 1 = 3 \]
and
\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x^2 - 1) = 4 \cdot 1^2 - 1 = 3, \]
\( \lim_{x \to 1} f(x) = 3 \). But \( f(1) \) is undefined and so \( x = 1 \) is a discontinuity. We can define
\[ f(x) = \lim_{x \to 1} f(x) = 3 \]
such that \( f(x) \) is continuous at \( x = 1 \). The answer is (D).

9. Given \( f(x) = \frac{x^3 - x}{x^2} \), \( f'(x) \) is equal to
(A) \( \frac{3x^2 - 1}{2x} \)  
(B) \( \frac{3x^2 - 1}{x^4} \)  
(C) \( \frac{(3x^2 - 2x)x^2 + 2x(x^3 - x)}{x^4} \)  
(D) \( 1 + \frac{1}{x^2} \)  
(E) \( \frac{x^2 - 1}{x^3} \)  
(F) None of These.

Solution: As \( \frac{x^3 - x}{x^2} = x - \frac{1}{x} \), \( f'(x) = 1 + \frac{1}{x^2} \). The answer is (D).

10. Given \( f(x) = x^4 - 3x^2 + 1 \), \( f''(x) \) is equal to
(A) \( 4x^3 - 3x \)  
(B) \( 12x^2 - 6x + 1 \)  
(C) \( 12x^3 - 6x \)  
(D) \( 12x^2 - 6x \)  
(E) \( 4x^2 - 3x \)  
(F) None of These.

Solution: \( f'(x) = 4x^3 - 6x \), \( f''(x) = 12x^2 - 6 \). The answer is (F).

11. Given the function \( f(x) = \frac{3x}{\sqrt{16 - x^2}} \), concerning the asymptotes of the graph \( y = f(x) \), which of the following statements is correct?
(A) \( y = f(x) \) has no vertical and horizontal asymptotes.
(B) \( y = f(x) \) has vertical asymptotes \( x = 4, x = -4 \) and horizontal asymptote \( y = 3 \).
(C) \( y = f(x) \) has vertical asymptotes \( x = 4, x = -4 \) and horizontal asymptote \( y = -3 \).
(D) \( y = f(x) \) has vertical asymptotes \( x = 4, x = -4 \) and no horizontal asymptotes.
(E) None of above is correct.

Solution: Set \( \sqrt{16 - x^2} = 0 \), we get \( x = \pm 4 \). And since \( \lim_{x \to 4^-} f(x) = \infty \) and \( \lim_{x \to -4^+} f(x) = -\infty \), \( x = \pm 4 \) are VAs. Since \( x \to \infty \), \( \sqrt{16 - x^2} \) is undefined and so \( \lim_{x \to \infty} \frac{3x}{\sqrt{16 - x^2}} \) does not exist. Hence there are no HAs. The answer is (D).

PART 2: This portion of the exam will be graded on a partial credit basis. **Answers without supporting work shown on the paper will receive NO credit.**

12. **(12 points)** For each of the following function \( f(x) \), find an equation of the line tangent to the curve \( y = f(x) \) at the indicated point \((a, f(a))\).

(a) \( f(x) = x \tan x, a = \pi \).

Solution: Since \( f'(x) = \tan x + x \sec^2 x \) (Product rule), the slope of the tangent line at \( a = \pi \) is \( m = f'(\pi) = \tan \pi + \pi \cdot \sec^2 \pi = 0 + \pi \cdot \frac{1}{\cos^2 \pi} = \pi \). Since \( f(\pi) = \pi \cdot \tan \pi = 0 \), the equation of the line tangent to the curve \( f(x) \) at \((a, f(a))\) is \( y - 0 = \pi (x - \pi) \), i.e., \( y = \pi (x - \pi) \).

(b) \( f(x) = \ln \left( \frac{x^2}{e^x} \right), a = 1 \).

Solution: Since \( f(x) = \ln(x^2) - \ln(e^x) = 2 \ln x - x, f'(x) = \frac{2}{x} - 1 \). So the slope of the tangent line at \( a = 1 \) is \( m = f'(1) = 2 - 1 = 1 \). Since \( f(1) = 2 \ln 1 - 1 = 0 - 1 = -1 \), the equation of the line tangent to the curve \( f(x) \) at \((a, f(a))\) is \( y - (-1) = 1 \cdot (x - 1) \), i.e., \( y = x - 2 \).

13. **(10 points)**

(a) Use your own language to state the Intermediate Value Theorem for continuous functions.

Solution: Suppose that \( f \) is continuous on the closed interval \([a, b]\) and \( W \) is any number between \( f(a) \) and \( f(b) \). Then, there is a number \( c \in (a, b) \) for which \( f(c) = W \).
(b) Find the domain of \( f(x) = \frac{x^4 - 2x - 2}{9 - x^2} - 1 \), and determine all discontinuities of
\[
f(x) = \frac{x^4 - 2x - 2}{9 - x^2} - 1
de this function (if there are any).

**Solution:** As the denominator can not be 0, \( 9 - x^2 \neq 0 \). And we get \( x \neq \pm 3 \). So the domain is \((-\infty, -3) \cup (-3, 3) \cup (3, \infty)\).

Since \( x = \pm 3 \) are not in the domain, they are discontinuities of \( f \).

(c) Use the Intermediate Value Theorem for continuous functions to explain why the equation \( x^4 - 2x - 2 \frac{9}{9} - x^2 - 1 = 0 \) must have a solution inside the interval \([0, 2]\).

**Solution:** \( f \) is continuous on \([a, b]\). As \( f(0) = \frac{0 - 0 - 2}{9} = \frac{-11}{9} < 0 \) and \( f(2) = \frac{2^4 - 2 \cdot 2 - 2}{9 - 4} - 1 = \frac{16 - 4 - 2}{5} = 2 > 0 \), by the intermediate value theorem, there exists a point \( c \in (0, 2) \) such that \( f(c) = 0 \). That means \( c \) is a solution of \( \frac{x^4 - 2x - 2}{9 - x^2} - 1 = 0 \).

14. *(12 points)* For each of the following function \( f(x) \), compute the derivatives, (do not simplify your answer).

(a) \( f(x) = \frac{x}{e^x} + x^2 \ln(x^5) \).

**Solution:** Since \( f(x) = \frac{x}{e^x} + x^2 \ln(x^5) = \frac{x}{e^x} + 5x^2 \ln x \), \( f'(x) = \frac{e^x - xe^x}{e^{2x}} + 5 \cdot 2x \ln x + 5x^2 \cdot \frac{1}{x} \).

(b) \( f(x) = e^x \sin x + \ln(x) \cos x \).

**Solution:** \( f'(x) = e^x \sin x + e^x \cos x + \frac{1}{x} \cos x + \ln x \cdot (- \sin x) \).

15. *(12 points)* Given a function

\[
f(x) = \frac{x^2 + 4}{x^2 - 4},
\]
do each of the following:

(a) Determine the domain and all discontinuities of \( f(x) \).

**Solution:** As the denominator can not be 0, \( x^2 - 4 \neq 0 \). And we get \( x \neq \pm 2 \). So the domain is \((-\infty, -2) \cup (-2, 2) \cup (2, \infty)\).

Since \( x = \pm 2 \) are not in the domain, they are discontinuities of \( f \).
(b) Find all horizontal and vertical asymptotes of the graph $y = f(x)$.

**Solution:** Set $x^2 - 4 = 0$, we get $x = \pm 2$. And since $\lim_{x \to 2} f(x) = \infty$ and $\lim_{x \to -2} f(x) = \infty$, $x = \pm 2$ are VAs.

Since $\lim_{x \to \infty} \frac{x^2 + 4}{x^2 - 4} = \lim_{x \to \infty} \frac{x^2(1 + \frac{4}{x^2})}{x^2(1 - \frac{4}{x^2})} = \lim_{x \to \infty} \frac{1 + \frac{4}{x^2}}{1 - \frac{4}{x^2}} = 1$. Hence $y = 1$ is a HA.

16. (10 points) Given $f(x) = \frac{1}{x + 3}$, use definition of derivative to find $f'(x)$. (No credit for answers not using the definition of a derivative).

**Solution:**

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} = \lim_{h \to 0} \frac{(x+3) - (x+h+3)}{h(x+h+3)(x+3)} = \lim_{h \to 0} \frac{-h}{h(x+h+3)(x+3)} = \lim_{h \to 0} \frac{-1}{(x+h+3)(x+3)} = \frac{-1}{(x+3)^2}.$$