Introduction to Combinatorial Optimization in Matroids

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1. Matroids and Examples

1. Independent Sets

(1.1.1) A matroid $M$ is an ordered pair $M = (E, \mathcal{I})$ consisting a finite set $E = E(M)$ and
a collection $\mathcal{I} = \mathcal{I}(M) \subseteq 2^E$ such that

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then there is some $e \in I_2$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Let $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ be two matroids. A map $f : E_1 \to E_2$ is a matroid
isomorphism if $f$ is a bijection such that for any $X \subseteq E$, $X \in \mathcal{I}_1$ if and only if $f(X) \in \mathcal{I}_2$. If
there is an isomorphism between $M_1$ and $M_2$, we write $M_1 \cong M_2$. Unless otherwise stated, we do not distinguish isomorphic matroids.

(1.1.2) (Graphic Matroids) Let $G$ be a graph with edge set $E = E(G) \neq \emptyset$. Let $\mathcal{I}$ denote the collection of subsets $I \in \mathcal{I}$ iff $I \subseteq E$ and $I$ induces an acyclic subgraph of $G$. This matroid is denoted by $M(G)$, called the cycle matroid of $G$. Any matroid isomorphic to an $M(G)$ is a graphic matroid.
(Note that if $M$ is graphic, one can always find a connected graph $G$ such that $M = M(G)$.)

(1.1.2) (Uniform Matroids) Let $n \geq r \geq 0$ be integers and let $E$ be a set of $n$ elements. Let $\mathcal{I}$ denote the collection of subsets of $E$ such that $I \in \mathcal{I}$ iff $|I| \leq r$. This matroid is denoted by $U_{r,n}$. Any matroid isomorphic to a $U_{r,n}$ is a uniform matroid.

(1.1.3) (Representable Matroids) Let $F$ be a field and let $A$ denote an $n \times m$ matrix with entries in $F$. Let $E$ denote the labelled columns of $A$, and $\mathcal{I}$ be the collection of subsets such that $I \in \mathcal{I}$ iff the columns vectors in $I$ are linearly independent over $F$. This matroid is denoted by $M_F[A]$, or simply $M[A]$. Any matroid isomorphic to an $M_F[A]$ is an $F$-representable matroid, or simply a representable matroid. A representable matroid is also called a matric matroid.

(1.1.4) Let $F = GF(p^m)$ be a finite field and $V(n, F) = V(n, p^m)$ denote the $n$-dimensional vector space over $GF(p^m)$.

(1.1.5) A single element $e \in E(M)$ is a loop if $\{e\} \notin \mathcal{I}(M)$. Two elements $e, e' \in E(M)$ are parallel in $M$ if neither $e$ nor $e'$ is a loop and $\{e, e'\} \notin \mathcal{I}(M)$.

(1.1.6) The matroids $M(K_{1,2} \cup K_{1,3}), M(T_5), U_{5,5}$ and $M[I_5]$ are isomorphic to each other, where $T_5$ denotes a tree on 5 vertices.

(1.1.7) The Affine matroid $AG(n,F)$.

(1.1.8) Exercise: Prove that $(E, \mathcal{I})$ is a matroid iff $\mathcal{I}$ satisfies (I2) and the following two conditions:

$$(\text{I1})' \quad \mathcal{I} \neq \emptyset.$$
(I3) If \( I_1 \) and \( I_2 \) are in \( \mathcal{I} \) and \(|I_2| = |I_1| + 1\), then there is an element \( e \in I_2 - I_1 \) such that \( I_1 \cup e \in \mathcal{I} \).

Exercise: Prove that \((E, \mathcal{I})\) is a matroid iff \( \mathcal{I} \) satisfies (I1), (I2) and the following:

(I3) If \( X \in E \) and \( I_1 \) and \( I_2 \) are maximal members of \( \{I : I \in \mathcal{I} \text{ and } I \subseteq X\} \), then \(|I_1| = |I_2|\).

Exercise: Let \( M_1 \) and \( M_2 \) be matroids on disjoint sets \( E_1 \) and \( E_2 \), respectively. Let
\[
E = E_1 \cup E_2 \text{ and } \mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), \text{ and } I_2 \in \mathcal{I}(M_2)\}.
\]
Prove that \((E, \mathcal{I})\) is a matroid (denoted by \( M_1 \oplus M_2 \), and called the direct sum of \( M_1 \) and \( M_2 \)).

Exercise: Let \( M_1 \) and \( M_2 \) be matroids on a set \( E \). Give an example to show that \((E, \mathcal{I}(M_1) \cap \mathcal{I}(M_2))\) need not be a matroid.

Exercise: Let \( M = (E, \mathcal{I}) \) be a matroid and let \( X \subseteq E \). Define
\[
\mathcal{I}(M|X) = \{I \in \mathcal{I}(M) : I \subseteq X\}.
\]
Show that \( \mathcal{I}(M|X) \) satisfies (I1)-(I3), and so \( M|X = (X, \mathcal{I}(M|X)) \) is a matroid, called the restriction of \( M \) to \( X \); or the deletion of \( E - X \) from \( M \).

2. Circuits

(Circuit Axioms):

(Part 1) Let \( M = (E, \mathcal{I}) \) be a matroid. Then the collection \( \mathcal{C} = (2^E - \mathcal{I})_{\text{min}} \) satisfies each of the following:

(C1) \( \emptyset \not\in \mathcal{C} \).

(C2) \( \mathcal{C} \) is a clutter.

(C3) If \( C_1 \neq C_2, C_1, C_2 \in \mathcal{C} \) and \( e \in C_1 \cap C_2 \), then there is a \( C_3 \in \mathcal{C} \) such that \( C_3 \subseteq C_1 \cup C_2 - \{e\} \). (Members in \( \mathcal{C} \) are circuits of \( M \). Write \( \mathcal{C} = \mathcal{C}(M) \) if \( M \) should be
indicated.)

(Part 2) Let \( C \subseteq 2^E \) be a collection satisfying (C1)-(C3). Then \( I = 2^E - C_{\text{upper}} \) satisfies (I1)-(I3), and so \( I = 2^E - C_{\text{upper}} \) is the collection of independent sets of a matroid \( M = (E, I) \). Furthermore, \( C(M) = C \).

**Proof** For (Part 1)(C3) Is \( C_1 \cup C_2 - e \in I(M) \)? If YES, then \( C_1 \cup C_2 \) has an independent set of cardinality \( |C_1 \cup C_2| - 1 \). \( \forall g \in C_1 - C_2, C_1 - g \in I(M) \). Choose \( I \) to be a maximal independent subset in \( C_1 \cup C_2 \) such that \( C_1 - g \subseteq I \). As \( C_2 \not\subseteq I \), \( \exists f \in C_2 - I \).

Thus \( |C_1 \cup C_2| - 1 \leq |I| \leq |C_1 \cup C_2 - \{g, f\}| = |C_1 \cup C_2| - 2 \).

(Part 2) (I3) Let \( I_1, I_2 \in I \) such that \( |I_1| < |I_2| \).

(1.2.2) Let \( G \) be a graph with \( E(G) \neq \emptyset \). The collection of all circuits of \( G \) satisfies (C1)-(C3), and so is the set of circuits of a matroid \( M \) on \( E(G) \). Note \( I(M) \) consists of the edge subsets which induce acyclic subgraphs. Therefore, \( M = M(G) \).

(1.2.3) \( C(M) = \emptyset \) iff \( M \cong U_{n,n} \).

(1.2.4) For \( n \geq r \geq 1 \), \( C(U_{r,n}) = \{X \in E : |X| = r + 1\} \).

(1.2.5) \( C(M) = \emptyset \) iff \( M \cong U_{n,n} \).

(1.2.6) Every graphic matroid is \( F \)-representable, for any field \( F \).

An algorithm to represent \( M(G) \) over a field \( F \):

(Step 1) For each vertex \( v_i \in V(G) \), choose a vector \( v_i \in F^m \), for some integer \( m \) large enough, so that all vectors \( v_1, v_2, \ldots, v_n \) are linearly independent. (One can choose \( m = n = |V(G)| \) and the \( v_i \)'s to be a standard basis of \( F^n \).)

(Step 2) Order \( V(G) \) linearly (Example: \( v_i < v_j \) iff \( i < j \)).

(Step 3) If \( e = v_i v_j \in E(G) \) and if \( v_j > v_i \), then define \( \phi(e) = v_j - v_i \).

Then \( \phi : E(G) \rightarrow F^m \) is an \( F \)-presentation of \( M(G) \) over \( F \).
(1.2.7) Proposition (1.1.5) can be restated as follows: An element \( e \in E(M) \) is a **loop** if \( \{e\} \in \mathcal{C}(M) \). Two elements \( e, e' \in E(M) \) are **parallel** if \( \{e, e'\} \in \mathcal{C}(M) \).

A matroid \( M \) that does not have a loop and parallel elements is a **simple matroid**, which is also called a **combinatorial geometry**.

(1.2.8) Exercise: Let \( C_1 \) and \( C_2 \) be circuits of a matroid \( M \) such that \( C_1 \cup C_2 = E(M) \) and \( C_1 - C_2 = \{e\} \). Prove that if \( C_3 \) is a circuit of \( M \), then either \( C_3 = C_1 \) or \( C_3 \supseteq C_2 - C_1 \).

(1.2.9) Exercise: (Strong Circuit Axiom) Let \( M = (E, \mathcal{I}) \) be a matroid with \( \mathcal{C} = \mathcal{C}(M) \). Then

\[(C3') \forall C_1, C_2 \in \mathcal{C}, \text{ if } e \in C_1 \cap C_2 \text{ and if } f \in C_2 - C_1, \text{ then } \exists C_3 \in \mathcal{C} \text{ such that } f \in C_3 \subseteq (C_1 \cup C_2) - e. (\text{Therefore, the circuit axioms } (C1)-(C3) \text{ are equivalent to the strong circuit axioms } (C1), (C2) \text{ and } (C3').)\]

(Hint: Choose a counterexample pair \( C_1, C_2 \) so that \( |C_1 \cap C_2| \) is minimized among all bad pairs.)

(1.2.10) Exercise: Let \( \mathcal{D} \) be a set of subsets of \( E \). Characterize when \( \mathcal{D} \) is the set of dependent sets of a matroid on \( E \).

(1.2.11) Exercise: Let \( I \in \mathcal{I}(M) \) and \( e \in E - I \), then either \( I \cup e \in \mathcal{I} \) or \( I \cup e \) contains exactly one member in \( \mathcal{C}(M) \).

(1.2.12) Exercise: Define a relation \( e \sim e' \) iff \( e = e' \) or if \( \exists C \in \mathcal{C}(M) \text{ such that } e, e' \in C \). Show that \( \sim \) is an equivalence relation.

3. Bases and the Rank Function

(1.3.1) (The Base Axiom):

(Part 1) Let \( M = (E, \mathcal{I}) \) be a matroid. Then \( \mathcal{B} = \mathcal{I}_{\text{max}} \) satisfies the following:

\[(B0) \text{ if } B_1, B_2 \in \mathcal{B}, \text{ then } |B_1| = |B_2|.\]

\[(B1) \mathcal{B} \text{ is non-empty.}\]
(B2) If \( B_1, B_2 \in \mathcal{B} \) and if \( x \in B_1 - B_2 \), then there is some \( y \in B_2 - B_1 \) such that \( (B_1 - x) \cup y \in \mathcal{B} \).

(1.3.2) Let \( B \in \mathcal{B}(M) \) and let \( e \in E - B \), then \( B \cup e \) contains exactly one circuit \( C(e, B) \), (called the \textbf{fundamental circuit of} \( e \) \textbf{with respect to} \( B \)).

(1.3.3) \((\text{B2}')\) If \( B_1, B_2 \in \mathcal{B} \) and if \( x \in B_1 - B_2 \), then there is some \( y \in B_2 - B_1 \) such that \( (B_2 - y) \cup x \in \mathcal{B} \).

(Members in \( \mathcal{B} \) are \textbf{bases} of \( M \). Write \( \mathcal{B} = \mathcal{B}(M) \) if \( M \) should be indicated.)

(1.3.4) \((\text{Part 2})\) Let \( \mathcal{B} \subseteq 2^E \) be a collection satisfying (B1)-(B2). Then \( \mathcal{I} = \mathcal{B}_{\text{lower}} \) satisfies (I1)-(I3), and so \( \mathcal{I} \) is a collection of independent sets of a matroid \( M = (E, \mathcal{I}) \). Furthermore, \( \mathcal{B} = \mathcal{B}(M) \).

(Part 3) A collection \( \mathcal{B} \subseteq 2^E \) is the collection of bases of a matroid on \( E \) if and only if \( \mathcal{B} \) satisfies (B1) and (B2').

(1.3.5) In \( M(G) \), a subset \( X \subseteq E(G) \) is a base if and only if \( X \) is a maximal spanning forest of \( G \); and if \( G \) is connected,

(1.3.6) Let \( C \in \mathcal{C}(M) \) and \( e \in C \), then \( \exists B \in \mathcal{B} \) such that \( C = C(e, B) \).

(1.3.7) \((\text{Rank Function})\):
Let \( M = (E, \mathcal{I}) \) be a matroid. Define a function \( r : 2^E \to \mathbb{Z}^+ \{0\} \), (called the \textbf{rank function} of \( M \), and sometimes denoted by \( r_M \) to emphasize \( M \)), by
\[
r(X) = \max\{|I| : I \subseteq X \text{ and } I \in \mathcal{I}\}.
\]
(We often write \( r(E) = r(M) \), and when \( r(M) = r \), we say that \( M \) is a rank-\( r \) matroid.) Then \( r \) satisfies the following:

(R1) For any \( X \subseteq E \), \( 0 \leq r(X) \leq |X| \).

(R2) (non-decreasing) If \( X \subseteq Y \subseteq E \), then \( r(X) \leq r(Y) \).

(R3) (submodularity) For any \( X, Y \in 2^E \),
\[
r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y).
\]
Let $E$ be a set and let $r$ be a function on $2^E$ satisfying (R1)-(R3). If $X, Y \subseteq E$ are such that $\forall y \in Y - X, r(X \cup y) = r(X)$, then $r(X) = r(X \cup Y)$.

(1.3.9) (Rank Axiom) Let $E$ be a nonempty set.

(Part 1) Let $r$ be an integer-valued function on $2^E$ satisfying (R2) and (R3). If $X, Y \subseteq E$ such that for all $y \in Y - X$, $r(X \cup y) = r(X)$, then $r(X \cup Y) = r(X)$.

(Part 2) Let $r$ be an integer-valued function on $2^E$ satisfying (R1)-(R3), and let $I(r) = \{X \subseteq E : r(X) = |X|\}$. Then $I(r)$ satisfying (I1)-(I3) and so $I(r)$ is the collection of independent sets of a matroid $M$. Moreover, $r_M$, the rank function of $M$, is $r$.

(1.3.10) Exercise: Let $A$ be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$  

For $q \in \{2, 3\}$, let $M_q[A]$ be the vector matroid of $A$ when $A$ is viewed over $GF(q)$. Show that:

(i) The sets of circuits of $M_2[A]$ and $M_3[A]$ are different.

(ii) $M_2[A]$ is graphic but $M_3[A]$ is not.

(iii) $M_2[A]$ is representable over $GF(3)$ but $M_3[A]$ is not representable over $GF(2)$.

(Hint: use linearly dependence of the binary space.)

(1.3.11) Exercise: Prove that $B$ is the collection of bases of a matroid on $E$ iff $B$ satisfies (B1) and the following two conditions:

(B2) If $B_1, B_2 \in B$ and $e \in B_1$, then there is an element $f \in B_2$ such that $(B_1 - e) \cup f \in B$.

(B3) If $B_1, B_2 \in B$ and $B_1 \subseteq B_2$, then $B_1 \subseteq B_2$.

(1.3.12) Exercise: Suppose that $X \subseteq Y \subseteq E$. Prove that if $E$ is the ground set of a matroid $M$ that has bases $B_1$ and $B_2$ with $X \subseteq B_1$ and $B_2 \subseteq Y$, then $M$ has a basis $B_3$ such that $X \subseteq B_3 \subseteq Y$.

(1.3.13) Exercise: Prove that the following statements are equivalent for an element $e$ of a matroid $M$:

(A) $e$ is a loop.
(B) $e$ is in no bases.
(C) $e$ is in no independent sets.

(1.3.14) Exercise:
(i) Prove that $U_{2,4}$ is representable over a field $F$ if and only if $|F| \geq 3$.
(ii) For $n \geq 2$, generalize (i) by finding necessary and sufficient condition for $U_{2,n}$ to be $F$-representable.
(iii) Prove the $U_{2,4}$ is neither graphic nor regular.

(1.3.15) Exercise: Suppose $B$ is a basis of a matroid $M$, $f \in E(M)$, and $e \in E(M) - B$. prove that $(B \cup e) - f$ is a basis of $M$ if and only if $f \in C(e, B)$.

(1.3.16) Exercise: Let $M$ be a matroid and $X$ and $Y$ be subsets of $E(M)$, each of which contains a basis. prove that if $|X| > |Y|$, then, for some element $x \in X - Y$, there is a basis contained in $X - x$.

(1.3.17) Exercise: Show that the collection $B$ of subsets of a set $E$ satisfying (B1), (B3) and the following condition need not be the set of bases of a matroid on $E$:

(B2) $^-If B_1 \in B$ and $e \in E - B_1$, then $\exists f$ such that $(B_1 - f) \cup e \in B$.

(1.3.18) Exercise: Find the values of $m$ and $n$ for which $U_{m,n}$ is graphic.

(1.3.19) Exercise:
(i) Give an example of two matroids $M_1$ and $M_2$ on a set $E$ such that a subset $X$ of $E$ is a circuit of $M_1$ if it is a basis of $M_2$.
(ii) Characterize all pairs $(M_1, M_2)$ of matroids for which the condition in (i) holds.

(1.3.20) Exercise: Prove that if $B_1$ and $B_2$ are bases of a matroid $M$ and $e \in B_1 - B_2$, then $\exists f \in B_2 - B_1$ such that $(B_1 - e) \cup f$ and $(B_2 - f) \cup e$ are bases of $M$.

(1.3.21) Exercise: Let $M$ be a matroid and $\mathcal{D}$ be the set of minimal subsets of $E(M)$ that meet every circuit of $M$. Prove that $\mathcal{D}$ is the set of bases of a matroid on $E(M)$.

(1.3.22) Exercise: Let $M$ be a matroid on $E$ and let $\mathcal{D}$ be defined as in (1.3.21). Show
that $D = \{ E - B : \forall B \in \mathcal{B}(M) \}$.

(1.3.23) Exercise: Let $X \subseteq E$. Then $X \in \mathcal{I}(M)$ iff $E - X$ meets every circuit.

(1.3.24) Exercise: Prove that a matroid $M$ is uniform iff it has no circuits of size less than $r(M) + 1$.

(1.3.25) Exercise: Let $f : 2^E \rightarrow \mathbb{Z} \cup \{0\}$ be a function such that

- (A) $f(\emptyset) = 0$;
- (B) if $X \subseteq Y \subseteq E$, then $f(X) \leq f(Y)$; and
- (C) if $X, Y \subseteq E$, then $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$.

Let $\mathcal{I}(f) = \{ I \subseteq E : f(X) \geq |X| \text{ for all } X \subseteq I \}$. Prove that $(E, \mathcal{I}(f))$ is a matroid, and find its rank function.

(1.3.26) Exercise:

(i) Find a matroid in which equality does not always hold in (R3).

(ii) Find all matroids in which equality does always hold in (R3).

(1.3.27) Exercise: Let $M$ be a matroid on the set $E$ and $k$ be a non-negative integer not exceeding $r(M)$. Define $r_{(k)} : 2^E \rightarrow \mathbb{Z} \cup \{0\}$ by $r_{(k)}(X) = \min\{k, r(X)\}$.

(i) Prove that $r_{(k)}$ is a rank function.

(ii) Find the collection of independent sets of the matroid in (i).

(1.3.28) Exercise: Let $M$ be a matroid on the set $E$ and $k$ be an integer with $r(M) \leq k \leq |E|$. Let $S_k$ denote the set of $k$-element subsets of $E$ having rank $r(M)$. Prove that $S_k$ is the set of bases of a matroid on $E$.

4. Closure and Closed Sets (Flats)

(1.4.1) (Closure Operator):

Let $M$ be a matroid with rank function $r$. Let $cl_M : 2^E \rightarrow 2^E$ be a set operator such that
for any \( X \in 2^E \),
\[
cl_M(X) = \{ x \in E : r(X \cup x) = r(X) \}.
\]

Then each of the following holds:

- (CL1) For any \( X \subseteq E \), \( X \subseteq cl_M(X) \).
- (CL2) If \( X \subseteq Y \subseteq E \), then \( cl_M(X) \subseteq cl_M(Y) \).
- (CL3) For any \( X \subseteq E \), \( cl_M(cl_M(X)) = cl_M(X) \).
- (CL4) If \( x \in E \), \( X \subseteq E \), and \( y \in cl_M(X \cup x) - cl_M(X) \), then \( x \in cl_M(X \cup y) \).

A subset \( X \subseteq E(M) \) is **closed** if \( cl_M(X) = X \). Closed sets are also called **flats**.

(1.4.2) Exercise: Determine the closure operator in graphic matroids, representable matroids and uniform matroids.

(1.4.3) Let \( M = (E, I) \) be a matroid with a closure operator \( cl_M \). Suppose \( X \subseteq E \) and \( x \in E \).

- (A) If \( X \in I \) and \( X \cup x \notin I \), then \( x \in cl_M(X) \).
- (B) If \( I \) is a maximal subset in \( I \), then \( cl_M(I) = cl_M(X) \).
- (C) \( cl_M(X \cup x) = cl_M(X) \) iff \( x \in cl_M(X) \).
- (D) \( cl_M(X) = X \cup \{ x \in E : \exists C \in C(M) \text{ such that } x \in C \subseteq X \cup x \} \).

(1.4.4) Let \( M \) be a matroid and \( X \) be a subset of \( E(M) \). Let \( \mathcal{A} \) denote the collection of subsets of \( E(M) \) such that \( Y \in \mathcal{A} \) if and only if for any \( y \in Y \), \( y \in cl_M(Y - y) \). Show that \( X \in \mathcal{C}(M) \) if and only if \( X \) is a minimal element in \( \mathcal{A} \).

(1.4.5) (Closure Axiom) Let \( E \) be a nonempty set. Let \( cl : 2^E \to 2^E \) satisfying (CL1)-(CL4). Let
\[
I = \{ X \subseteq E : x \notin cl(X - x) \text{ for all } x \in X \}.
\]
Then \( I \) satisfies (I1)-(I3), and so there is a matroid \( M = (E, I) \). Moreover, \( cl = cl_M \).

(1.4.6) Exercise: Let \( M \) be a matroid and \( r \) and \( cl \) be its rank function and its closure operator. Prove the following:

- (0) If \( B_X \in \mathcal{B}(M|X) \), then \( cl(X) = cl(B_X) \), and so \( B_X \in \mathcal{B}(M|cl(X)) \).
- (i) If \( X \subseteq cl(Y) \) and \( cl(Y) \subseteq cl(X) \), then \( cl(X) = cl(Y) \).
(ii) If \( Y \subseteq \text{cl}(X) \), then \( \text{cl}(X \cup Y) = \text{cl}(X) \).

(iii) The intersection of all of the flats containing \( X \) equals \( \text{cl}(X) \).

(iv) \( r(X \cup Y) = r(X \cup \text{cl}(Y)) = r(\text{cl}(X) \cup \text{cl}(Y)) = r(\text{cl}(X \cup Y)) \).

(v) If \( X \subseteq Y \) and \( r(X) = r(Y) \), then \( \text{cl}(X) = \text{cl}(Y) \).

(1.4.7) Exercise: (Spanning Sets and Hyperplanes) Let \( M = (E, \mathcal{I}) \) be a matroid. The **spanning sets** of \( M \) are the elements in \( \mathcal{S}(M) = \{ X \subseteq E | r(X) = r(M) \} \) and the **hyperplanes of** \( M \) are the elements in \( \mathcal{H}(M) = \{ X \subseteq E | \text{cl}_M(X) = X \text{ and } r(X) = r(M) - 1 \} \).

Let \( X \subseteq E \). Each of the following holds:

(A) \( X \in \mathcal{S}(M) \) iff \( X \) contains a base \( B \in \mathcal{B}(M) \).

(B) \( X \in \mathcal{H}(M) \) iff \( X \) is a maximal nonspanning set. (\( i.e. \mathcal{H}(M) = (\mathcal{S}(M)^c)^o \).)

(C) \( X \in \mathcal{B}(M) \) iff \( X \) is a minimal spanning set. (\( i.e. \mathcal{B}(M) = \mathcal{S}(M)_o \).

(1.4.8) Exercise: Prove that statements (a)-(g) below are equivalent for an element \( e \) of a matroid \( M \):

(a) \( e \) is in every basis.

(b) \( e \) is in no circuits.

(c) If \( X \subseteq E(M) \) and \( e \in \text{cl}(X) \), then \( e \in X \).

(d) \( r(E(M) - e) = r(E(M)) - 1 \).

(e) \( E(M) - e \) is a flat.

(f) \( E(M) - e \) is a hyperplane.

(g) If \( I \) is an independent set, then so is \( I \cup e \).

(1.4.9) Exercise: Let \( X \) and \( Y \) be flats of a matroid \( M \) such that \( Y \subseteq X \) and \( r(Y) = r(X) - 1 \). Prove that \( M \) has a hyperplane \( H \) such that \( Y = H \cap X \).

(1.4.10) Exercise: Let \( X \) be a subset of a matroid \( M \). Prove that \( X \) is a hyperplane iff \( E(M) - X \) is a minimal set intersecting every basis.

(1.4.11) Exercise: Let \( E \) be a set. A function \( r : 2^E \rightarrow \mathbb{Z} \cup \{0\} \) is the rank function of a matroid \( M \) if and only if each of the following holds:

\( (R1)' \) \( r(\emptyset) = 0 \).

\( (R2)' \) If \( X \subseteq E \) and \( x \in E \), then \( r(X) \leq r(X \cup x) \leq r(X) + 1 \).

\( (R3)' \) If \( X \subseteq E \) and \( x, y \in E \) such that \( r(X \cup x) = r(X \cup y) = r(X) \), then \( r(X \cup \{x, y\}) = \)
r(X).

(1.4.12) Exercise: Let $S$ be a collection of subsets of a set $E$. Prove that $S$ is the set of spanning sets of a matroid on $E$ if the following hold:

(S1) $S \neq \emptyset$.

(S2) If $S_1 \in S$ and $S_1 \subset S_2$, then $S_2 \in S$.

(S3) If $S_1, S_2 \in S$ and $|S_1| > |S_2|$, then $\exists e \in S_1 - S_2$ such that $S_1 - e \in S$.

(1.4.13) Exercise: Let $F$ be a collection of subsets of a set $E$. Prove that $F$ is the set of flats of a matroid $M$ if the following hold:

(F1) $E \in F$.

(F2) If $F_1, F_2 \in F$, then $F_1 \cap F_2 \in F$.

(F3) If $F \in F$ and $\{F_1, \cdots, F_k\}$ is the set of minimal members of $F$ that properly contain $F$, then the sets $F_1 - F, F_2 - F, \cdots, F_k - F$ partitions $E - F$.

(1.4.14) Exercise: Let $C_1, C_2, \cdots, C_n$ be distinct circuits of a matroid $M$ such that, for all $j \in \{1, 2, \cdots, n\}$, $C_j \not\subseteq \cup_{i \neq j} C_i$. Prove that if $D \subseteq E = E(M)$ and $|D| < n$, then $M$ has a circuit $C$ such that $C \subseteq (\cup_{i=1}^k C_i) - D$.

(1.4.15) Exercise: If $X$ is a flat in a matroid $M$, and if $I \in B(M|X)$, then $X = cl(I)$. 

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2. Greedy Algorithm and Matroids

1. Greedy Algorithm

(2.1.1) Let \( M = (E, \mathcal{I}) \) be a matroid and let \( c : E \rightarrow \mathbb{R} \) be a weight function. The **Maximum Weight Independent Set Problem** seek an independent subset \( J \in \mathcal{I}(M) \) such that
\[
c(J) := \sum_{e \in J} c(e) = \max\{c(I) : I \in \mathcal{I}(M)\}.
\]

(2.1.2) **Greedy Algorithm** for the Maximum Weight Independent Set Problem in a given matroid \( M = (E, \mathcal{I}) \) with a weight function \( c \):

(Step 1) Label the elements in \( E = \{e_1, e_2, \ldots, e_m\} \) so that
\[
c(e_1) \geq c(e_2) \geq \cdots \geq c(e_h) \geq 0 \geq c(e_{h+1}) \geq \cdots \geq c(e_m).
\]

(Step 2) Set \( J := \emptyset \).

(Step 3) For \( i = 1 \) to \( h \), if \( J \cup \{e_i\} \in \mathcal{I}(M) \), then set \( J := J \cup \{e_i\} \).

Remark: Suppose that there is an subroutine with running time not to exceed a polynomial of \( |E| \) which determines if a given subset of \( E \) is in \( \mathcal{I}(M) \) or not in \( \mathcal{I}(M) \). Then the running time for executing Algorithm (2.1.2) is polynomial.

(2.1.3) Algorithm (2.1.2) can determine an optimal solution for Problem (2.1.1). Moreover, if \( \forall e_i, e_j \) with \( i \neq j \), \( c(e_i) \neq c(e_j) \), then the optimal solution for Problem (2.1.2) is unique.

Proof: Let \( J = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \) be a solution obtained by running Algorithm (2.1.2). Let \( J' = \{e_{j_1}, e_{j_2}, \ldots, e_{j_l}\} \in \mathcal{I}(M) \) satisfying \( c(J') \geq c(J) \). We shall prove that \( c(J') = c(J) \) (and leave the uniqueness part as an exercise).

Since \( J \) is obtained by running Algorithm (2.1.2), we may assume that \( e_{i_l} \) is the \( l \)th element added to \( J \) in Algorithm (2.1.2). Thus
\[
c(e_{i_l}) \geq c(e_{i_2}) \geq \cdots \geq c(e_{i_k}).
\]

We may also assume that \( J' \) satisfies
\[
c(e_{j_1}) \geq c(e_{j_2}) \geq \cdots \geq c(e_{j_l}).
\]
If \( c(J') > c(J) \), then let \( s \geq 0 \) be the smallest integer such that either \( s \leq k \) and \( c(e_{j_s}) > c(e_{i_s}) \), or \( s > k \) and \( c(j_s) > 0 \).

Let \( I = \{e_{i_1}, e_{i_2}, \ldots, e_{i_{s-1}}\} \), and \( X = I \cup \{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\} \). By Step 3 of Algorithm (8.2.2), \( I \in \mathcal{I}(M|X) \). Since \( c(e_{j_1}) \geq c(e_{j_2}) \geq \cdots \geq c(e_{j_s}) \) and by the choice of \( s \), for each \( l \in \{1, 2, \cdots, s\} \), if \( e_l \notin I \), then \( I \cup e_l \notin \mathcal{I}(M) \). Thus \( r(M|X) = s - 1 \) contrary to the fact that \( \{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\} \), being a subset of an independent set \( J' \), is also independent in \( M|X \).

(2.1.4) **Notation:** Let \( J \subseteq E \) be a subset. The **characteristic function** of \( J \) is

\[
x_J(e) = \begin{cases} 
1 & \text{if } e \in J \\
0 & \text{if } e \notin J 
\end{cases}
\]

For any function \( f : E \rightarrow \mathbb{R} \), and for any subset \( T \subseteq E \), denote

\[
f(T) = \sum_{e \in T} f(e).
\]

(2.1.5) (Another proof) Let \( J \in \mathcal{I}(M) \) be a subset obtained by running Algorithm (2.1.2), and let \( J' \in \mathcal{I}(M) \) be an optimal solution to Problem (2.1.1). Using the notation in Step 1 of Algorithm (2.1.2), for each \( i = 1, 2, \ldots, s \), define \( T_0 = \emptyset \), and \( T_i = \{e_1, e_2, \cdots, e_i\} \). By Step 3 of (2.1.2), \( J \cap T_i \) is a maximal independent subset in \( T_i \). Thus

(i) for each \( i \), \( x_J(T_i) \geq x_{J'}(T_i) \).

(ii) Moreover,

\[
c(J') = \sum_{t=1}^{h} c(e_t)x_{J'}(e_t) + \sum_{t=h+1}^{m} c(e_t)x_{J'}(e_t)
= \sum_{t=1}^{h} c(e_t)(x_{J'}(T_t) - x_{J'}(I_{t-1})) + \sum_{t=h+1}^{m} c(e_t)x_{J'}(e_t)
= \sum_{t=1}^{h-1} (c(e_t) - c(e_{i+1}))x_{J'}(T_t) + c(e_h)x_{J'}(e_h) + x_{J'}(I_{i-1}) + \sum_{t=h+1}^{m} c(e_t)x_{J'}(e_t)
\leq \sum_{t=1}^{h-1} (c(e_t) - c(e_{i+1}))x_{J}(T_t) + c(e_h)x_{J'}(e_h) + x_{J}(I_{i-1}) + \sum_{t=h+1}^{m} c(e_t)x_{J}(e_t)
= c(J).
\]

(2.1.6) Let \( M \) be a matroid with a non negative weight function \( c \). For each \( I \in \mathcal{I}(M) \), we
label the elements in $I$ so that if $I = \{a_1, a_2, \cdots, a_s\}$, then
\[ c(a_1) \geq c(a_2) \geq \cdots \geq c(a_s). \]

For $I_1, I_2 \in \mathcal{I}(M)$ such that $I_1 = \{a_1, a_2, \cdots, a_s\}$ and $I_2 = \{b_1, b_2, \cdots, b_t\}$, if there exists an index $k$ such that $c(a_i) = c(b_i)$ for all $i$ with $1 \leq i \leq k$ but $c(a_k) > c(b_k)$; or if for all $1 \leq i \leq t$, $c(a_i) = c(b_i)$ but $s > t$, then we say that $I_1$ is lexicographically greater than $I_2$.

For an independent set $I_0 \in \mathcal{I}(M)$, if $\forall I \in \mathcal{I}(M)$, $I$ is not lexicographically greater than $I_0$, then $I_0$ is a lexicographically maximal element. As every lexicographically maximal element must be a basis of $M$, a lexicographically maximal element is also called a lexicographically maximal basis. If a basis $B_0$ satisfies the property that for all basis $B$ in $M$, $c(B_0) \geq c(B)$. Then $B_0$ is a maximum weight basis.

(2.1.7) Observation: If $c$ is a non negative weight function, then Problem (2.1.1) seeks a maximum basis. Algorithm (2.1.2) outputs a lexicographically maximal basis.

(2.1.8) (Edmonds and Rota) Let $\mathcal{I}$ be a set of subsets of a set $E$ satisfying Independent Axioms (I1) and (I2) (in such case the pair $(E, \mathcal{I})$ is called an independent system). Then each of the following holds.

(i) If $M = (E, \mathcal{I})$ is also a matroid, then for any non negative weigh function $c : E \mapsto \mathbb{R}$, every lexicographically maximal element is also a maximum basis.

(ii) If (2.1.8)(i) holds, then $\mathcal{I}$ also satisfies Axiom (I3)”, and so $(E, \mathcal{I})$ must be a matroid.

Proof: (i) Suppose that $M = (E, \mathcal{I})$ is also a matroid. Fix a weight function $c$. Let $B = \{b_1, b_2, \cdots, b_r\}$ be a lexicographically maximal element of $M$. To show that $B$ is a maximum basis of $M$, it suffices to show that for any independent set $I \in \mathcal{I}(M)$, we always have $c(B) \geq c(I)$.

We may assume that both $B = \{b_1, b_2, \cdots, b_r\}$ and $I = \{a_1, a_2, \cdots, a_s\}$ are so ladled such that
\[ c(b_1) \geq c(b_2) \geq \cdots \geq c(b_r) \text{ and } c(a_1) \geq c(a_2) \geq \cdots \geq c(a_s). \]

If for some $k$, we have $c(a_k) > c(b_k)$, then consider $B_{k-1} = \{b_1, b_2, \cdots, b_{k-1}\}$ and $I_k = \{a_1, a_2, \cdots, a_k\}$. Since $\mathcal{I}(M)$ satisfies Axioms (I1)-(I3), and by (I3), there exists $a_i \in I_k - B_{k-1}$ such that $B_{k-1} \cup a_i \in \mathcal{I}(M)$. Since $c(a_i) \geq c(a_k) > c(b_k)$, it follows by Definition (2.1.6) that $B_{k-1} \cup a_i$ is lexicographically greater than $B$, contrary to the assumption.
that $B$ is a lexicographically maximal element of $M$. Similarly, if $s > r$, then there must be an $a_i$ with $1 \leq i \leq r+1$ such that $B \cup a_i \in \mathcal{I}(M)$, also contrary to the assumption that $B$ is a lexicographically maximal element of $M$. Therefore, $r \geq s$ and for each $i \in \{1, 2, \cdots, s\}$, $c(b_i) \geq c(a_i)$. It follows that $c(B) \geq c(I)$, as desired.

(ii) Suppose that $(I3)$ fail. Then there must be a subset $X \subseteq X$ together with subsets $I, I' \subseteq X$, such that $I, I' \in \mathcal{B}(M|X)$ but $|I| < |I'|$. Pick a real number $\epsilon > 0$ and define $c : E \mapsto \mathbb{R}$ as follows, for each $e \in E$,

$$c(e) = \begin{cases} 
1 + \epsilon & \text{if } e \in I \\
1 & \text{if } e \in I' - I \\
0 & \text{if } e \in E - (I \cup I')
\end{cases}.$$ 

Then by Definition (2.1.6), $I$ is a lexicographically maximal element. Compute

$$c(I') = |I'| - |I| + |I' \cap I|(1 + \epsilon) = |I'| + |I' \cap I|\epsilon,$$
and $c(I) = |I| + |I|\epsilon$.

To make $c(I) < c(I')$ (so that we obtain a contradiction), we need

$$|I'| + |I' \cap I|\epsilon > |I| + |I|\epsilon \iff |I'| - |I| \geq (|I| - |I' \cap I|)\epsilon \iff \epsilon \leq \frac{|I'| - |I|}{|I' - I|}.$$ 

Note that as $I \in \mathcal{B}(M|X)$, we must have $I - I' \neq \emptyset$, and so the right hand side is finite. Choose $\epsilon = \frac{|I'| - |I|}{2|I|} < \frac{|I'| - |I|}{|I' - I|}$. Then we have $c(I) < c(I')$ and so $I$ is lexicographically maximal but not maximum, contrary to the assumption that (2.1.8)(i) holds. Therefore, $(I3)$ must hold and so $(E, \mathcal{I})$ must be a matroid.

(2.1.9) Exercise: Let be a matroid and let $c : E \mapsto \mathbb{R}$ is a nonnegative weight function. For a fixed subset $\subseteq E(G)$, the following are equivalent.

(i) $B$ is a maximum basis.

(ii) $B$ is a lexicographically maximal element.

(iii) For each $e \in B$, define $I = \{e' \in B|c(e') > c(e)\}$ and $X = e' \in E|c(e') > c(e)\}$. Then $I \in \mathcal{B}(M|X)$.
3. Duality and Minors

1. Duality

(3.1.1) Let $M = (E, I)$ be a matroid with bases $B = B(M)$. Then $B^c$ also satisfies (B1)-(B2). (Thus there is a matroid on $E$ with bases $B^c$. This matroid is denoted by $M^*$, called the dual of $M$. Note that $B^c = B(M^*)$.)

(3.1.2) $(M^*)^* = M$.

(3.1.3) $U_{r,n}^* = U_{n-r,n}$, for any $0 \leq r \leq n$.

(3.1.4) Members in $B(M^*)$ are called cobases of $M$. Independent sets in $M^*$ are coindependent in $M$. Similarly we define cocircuits, corank, ... of $M$. (Because of this, the independent sets, bases, circuits ... of $M^*$ are also denoted $I^*(M), B^*(M), C^*(M), ...$, respectively.) Then
(i) $I^* \in I(M^*) \iff E - I^* \in S(M)$.
(ii) $C^* \in C(M^*) \iff E - C^* \in H(M)$.

Proof (ii) $\forall e \in C^*, C^* - e \in I(M^*)$ and so $E - (C^* - e) \in S(M)$. But $E - C^* \not\in H(M)$ since $C^* \not\in I(M^*)$. To see $H = E - C^*$ is a flat, one can show that $\forall e \not\in H, e \not\in cl_M(H)$.

(3.1.5) Let $G$ be a graph. If $M'$ denote the bond matroid of $G$. Then $(M(G))^* = M'$. We often write $M^*(G)$ for $(M(G))^*$. (A matroid $M$ is called a cographic matroid if it is isomorphic to a bond matroid of a graph $G$).

Proof Apply (3.1.4)(ii).

(3.1.6) If $M$ is isomorphic to $M^*$, then $M$ is self-dual. As an example, $M(K_4)$ is self-dual. If $I(M) = I(M^*)$, then $M$ is identically self-dual. As an example, $U_{r,2r}$ is identically self-dual. Also, $R_8$ is identically self-dual.

(3.1.7) Exercise: Let $X \subseteq E(M)$. Then $X$ is a cocircuit of $M$ if and only if $X$ satisfies the
property:
\[ \forall B \in \mathcal{B}(M), \; B \cap X \neq \emptyset, \]
and no proper subset of \( X \) satisfies this property.

(3.1.9) Exercise: A subset \( X \subseteq E \) is a circuit-hyperplane of \( M \) if and only if \( E - X \) is a circuit-hyperplane of \( M^* \).

(3.1.10) Exercise: Let \( X \subseteq E \) be a circuit-hyperplane of \( M \). Then \( \mathcal{B}(M) \cup \{X\} \) is the set of bases of a matroid \( M' \) on \( E \). (The matroid \( M' \) is called a relaxation of \( M \). We also say that \( M' \) is obtained from \( M \) by relaxing a circuit-hyperplane \( X \).)

(3.1.11) Let \( M = (M, \mathcal{I}) \) be a matroid with rank \( r \), and let \( r^* \) denote the rank of \( M^* \).

(i) Let \( I \in \mathcal{I}(M) \) and \( I^* \in \mathcal{I}(M^*) \) be disjoint subsets of \( E(M) \). Then there exist \( B \in \mathcal{B}(M) \) and \( B^* \in \mathcal{B}(M^*) \) such that

\[ I \subseteq B, \; I^* \subseteq B^* \quad \text{and} \quad B \cap B^* = \emptyset. \]

(ii) For any \( X \subseteq E \), if \( I^* \in \mathcal{B}(M^*|X) \) and if \( I = X - I^* \), then the \( B \) and \( B^* \) in (i) satisfy that \( r(E - X) = |B - X| \).

(iii) \( r^*(X) = |X| - r(E) + r(E - X) \).

**Proof:**

(i) Since \( I^* \in \mathcal{I}(M^*) \iff E - I^* \in \mathcal{S}(M) \), and so \( \exists B \in \mathcal{B}(M - I^*) \cap \mathcal{B}(M) \) with \( I \subseteq B \). Thus \( B^* = E - B \).

(ii). Choose \( I^* \in \mathcal{B}(M^*|X) \) and \( I = X - I^* \). By (i), \( \exists B \in \mathcal{B}(M) \) and \( B^* \in \mathcal{B}(M^*) \) such that \( B \cap B^* = \emptyset, \; I \subseteq B \) and \( I^* \subseteq B^* \). Note that \( r(X) \geq |I| = |B \cap X| \) and \( r(E - X) \geq |B - X| \). There exist \( I_1 \in \mathcal{B}(M - X) \), and \( B_1 \in \mathcal{B}(M) \cap 2^{E-I^*} \) such that \( B - X \subseteq I_1 \subseteq B_1 \). If \( B_1 \cap X \neq I \), then \( \exists e \in I - B_1 \subseteq E - B_1 \in \mathcal{B}(M^*) \) with \( I^* \subseteq B^* \), contrary to \( I^* \in \mathcal{B}(M^*|X) \). Hence \( B_1 \cap X = I \), and so

\[ |B| = |B \cap X| + |B - X| \leq |I| + r(E - X) = |I| + |B_1 - X| = |B_1|, \]

which implies \( r(E - X) = |B - X| \).

(iii). Now (iii) follows by \( I^* = B^* \cap X \), (ii) and by

\[ |X| + r(E - X) = |B^* \cap X| + |B \cap X| + |B - X| = |B| + |B^* \cap X|. \]
Another way to directly prove (iii):
Let \( I^* \in \mathcal{B}(M^*|X) \) and \( I \in \mathcal{B}(M - X) \). Then there exist \( B \in \mathcal{B}(M) \) and \( B^* \in \mathcal{B}(M^*) \) such that \( B \cap B^* = \emptyset \) and \( I \subseteq B \) and \( I^* \subseteq B^* \). Note that

\[ I = B \cap (E - X), |I| = r(E - X), I^* = B^* \cap X, |I^*| = r^*(X). \]

Thus

\[ r(M) = |B| = r(E - X) + |B \cap X| \]
\[ |I^*| = |B^* \cap X| \]

Adding these side by side and applying \( E = B \cap B^* \) yield (iii).

(3.1.12) Exercise: Let \( M \) be a matroid. Then

(i) \( \mathcal{B}(M)^c = \mathcal{B}(M^*) \).

(ii) \( \mathcal{H}(M)^c = \mathcal{C}(M^*) \).

(iii) \( \mathcal{C}(M)^c = \mathcal{H}(M^*) \).

(3.1.13) Let \( M = (E, \mathcal{I}) \) be a matroid. Then both (also their dual form)

\[ \mathcal{C}(M^*) = b(\mathcal{B}(M)), \text{ and } b(\mathcal{C}(M^*)) = \mathcal{B}(M). \]

**Proof:** it suffices to prove \( \mathcal{C}(M^*) = b(\mathcal{B}(M)) \), which is equivalent to that every codependent set of \( M \) has nonempty intersection with every basis of \( M \).

Alternatively, (A) \( \iff (B) \iff (C) \iff (D) \iff (E): \)

(A) \( C^* \in \mathcal{C}(M^*) \). (by (2.1.1C)(ii))

(B) \( E - C^* \in \mathcal{H}(M) \). (by def. of \( \mathcal{H}(M) \))

(C) \( E - C^* \in \mathcal{S}(M) \) but \( \forall e \in C^*, E - (C^* - e) \) contains a basis of \( M \).

(D) \( C^* \cap B \neq \emptyset, \forall B \in \mathcal{B}(M) \), but \( \forall e \in C^*, \exists B \in \mathcal{B}(M) \) such that \( (C^* - e) \cap B = \emptyset \).

(E) \( C^* \in b(\mathcal{B}(M)) \).

(3.1.14) If \( C \in \mathcal{C}(M) \) and \( C^* \in \mathcal{C}(M^*) \), then \( |C \cap C^*| \neq 1 \).

**Proof:** Let \( H - E - C^* \in \mathcal{H}(M) \). If \( C \cap C^* = x \), then \( x \in C \subseteq H \cup x \) and so \( x \in \text{cl}(H) - H \),
and so $H$ is not closed.

(3.1.15) Let $M = (E, T)$ be a matroid and let $D \subseteq E$ be a subset. Then $D \in C(M)$ iff $D$ satisfies each of the following.

(i) $D \neq \emptyset$.

(ii) $|D \cup C^*| \neq 1$, $\forall C^* \in C(M^*)$.

(iii) No proper subset of $D$ satisfies both (i) and (ii).

Proof:  (Sufficiency) It suffices to show that $D$ is dependent. If not, then augment $D$ to a basis $B \in B(M)$. Then use fundamental circuits in $M^*$.

(Necessity) Let $D \in C(M)$. By (2.1.7), $b(C(M)) = B(M^*)$. Thus $\forall B^* \in B(M^*)$, $D \cap B^* \neq \emptyset$.

For any $C^* \in C(M^*)$, if $C^* \cap D = \{e\}$, then there exist $B \in B(M)$ and $B(M^*)$ such that $B \cap B^* = \emptyset$ and $D - e \subseteq B$ and $C^* - e \subseteq B^*$. But $e \in E = B \cap B^*$, absurd.

Let $D_x = D - x$, $\forall e \in D$. Assume by contradiction that $D_x$ satisfies both (i) and (ii). Then by the Sufficiency, $D_x$ is dependent, absurd.

2. Examples of Dual Matroids

(3.2.1) The Dual of Representable Matroids Let $A$ be an $m \times n$ matrix over a field $F$ with rank $r > 0$. Then there is a $r \times (n - r)$ matrix $D$ such that $M_F[A] \cong M_F[I_r|D]$. The matrix $[I_r|D]$ is called the standard representative matrix for $M$.

(3.2.2) Exercise: If $M = M[I_r|D]$, then $M^* = M[-D^T|I_{n-r}]$.

(3.2.3) Exercise: A matroid $M$ is $F$-representable if and only if $M^*$ is $F$-representable.

(3.2.4) Exercise: For a matrix $A$ over a field $F$, $\mathcal{R}(A)$ denotes the row space of $A$, the subspace of $F^n$ spanned by the row vectors of $A$.

Let $[I_r|D]$ be an $r \times n$ matrix over a field $F$, where $1 \leq r \leq n - 1$. Then the orthogonal subspace of $\mathcal{R}([I_r|D])$ is $\mathcal{R}[-D^T|I_r]$.  

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(3.2.5) \( F_7^* = M_{GF(2)}[I_4|D] \), where
\[
D = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]

(3.2.6) None of \( F_7, U_{2,4}, M^*(K_5), M^*(K_{3,3}) \) is graphic.

**Proof:** This is an application of the rank function. Suppose that \( M^*(K_{3,3}) \cong M(G) \) for some connected \( G \). Note that \(|E(K_{3,3})| = 9\) and \( r(M^*(K_{3,3})) = 5\). By (2.1.5)(iii), \( r(M) = 9 - 5 = 4\), and so \( G \) has 5 vertices and 9 edges. The average degree of \( G \) is \( 18/5 < 4\), and so \( G \) must have a vertex of degree 3. By (3.1.5), \( M^*(K_{3,3}) \) has a 3-cycle.

(3.2.7) Let \( G \) be a plane graph (a planar graph with a specific embedding on the plane). The **geometric dual of** \( G \), denoted \( G^* \), is obtained as follows: Choose a single vertex \( v_F \) in each face \( F \) of \( G \). These vertices will be the vertices of \( G^* \). If the edges \( \{e_1, \ldots, e_k\} \) are common to the boundaries of two faces \( F \) and \( F' \), the join \( v_F \) and \( v_{F'} \) by edges \( e'_1, \ldots, e'_k \) in \( G^* \), where \( e'_i \) crosses \( e_i \) but no other edges of \( G \). The only vertex common to two distinct edges of \( G^* \) can be their ends. If an edge \( e \) lies on the boundary of a single face \( F \) of \( G \), then a loop \( e' \) is added to \( v_F \) that crosses \( e \) but no other edge of \( G \) or \( G^* \). A geometric dual of a planar graph \( G \) is a geometric dual of an planar embedding of \( G \).

(3.2.8) Exercise: Let \( G \) be a graph. A graph \( G' \) is an **abstract dual** of \( G \) if there exists a bijection \( \phi : E(G) \mapsto E(G') \) such that \( X \in C(M(G)) \) iff \( \phi(X) \) is a bond of \( G' \).

(3.2.9) Every geometric dual of \( G \) is an abstract dual. It is possible that a planar graph may have two non isomorphic geometric duals.

(3.2.10) If \( G^* \) is a geometric dual of \( G \), then
\[
M^*(G)(= M(G^*)) \cong M(G^*).
\]

**Proof:** Let \( G_0 \) be a planar embedding of \( G \) and \( G^* \) be the geometric dual of \( G_0 \). Let \( \alpha : E(G_0) \mapsto E(G^*) \) be the identity map in constructing \( G^* \) from \( G_0 \). By (3.1.5), it suffices
to show these two claims.

**Claim 1** If \( C \in \mathcal{C}(M(G_0)) \), then \( \alpha(C) \) is a bond of \( G^* \). Use Jordan curve in the plane to see that \( \alpha(C) \) is an edge cut in \( G^* \). What does it mean if it is not minimal?

**Claim 2** If \( D \) is a bond of \( G^* \), then \( \alpha^{-1}(D) \in \mathcal{C}(M(G_0)) \).

(3.2.11) Exercise: Find each of the following:
- (1A) all self-dual uniform matroids;
- (1B) all identically self-dual uniform matroids;
- (1C) all self-dual graphic matroids on six or fewer elements;
- (1D) all identically self-dual graphic matroid on six or fewer elements;
- (1E) an infinite family of simple graphic self-dual matroids.

(3.2.12) Exercise: Let \( M \) be a matroid. Show that \( M^* \) has two disjoint circuits iff \( M \) has two hyperplanes whose union is \( E(M) \).

(3.2.13) Exercise: Let \((X, Y)\) be a partition of the ground set \( E \) of a matroid \( M \) such that \( X \in \mathcal{I}(M) \) and \( Y \in \mathcal{I}(M^*) \). Show that \( X \in \mathcal{B}(M) \) and \( Y \in \mathcal{B}(M^*) \).

(3.2.14) Exercise: For each of the following properties, determine whether the property is necessary or sufficient (or both) for \( M \) to be identically self-dual:
- (A) Every basis of \( M \) is a cobasis of \( M \).
- (B) Every flat of \( M \) is a coflat of \( M \).
- (C) The flats of \( M \) and \( M^* \) coincide.
- (D) Every circuit of \( M \) is a cocircuit of \( M \).

(3.2.15) Exercise: Let \( e \) and \( f \) be distinct elements of a matroid \( M \). Prove that every circuit containing \( e \) also containing \( f \) iff \( \{e\} \) or \( \{e, f\} \) is a cocircuit.

(3.2.16) Exercise: Let \( n \) be a positive integer. Show that the number of non-isomorphic matroids on an \( n \)-element set is at most twice the number of non-isomorphic self-dual matroids on a \( (2n) \)-element set.

(3.2.17) Exercise: Prove that a matroid is uniform iff every circuit meets every cocircuit.
(3.2.18) Exercise: Is this true: \( M \) is uniform if and only if every hyperplane of \( M \) is independent?

(3.2.19) Exercise: Let \( M \) be a rank-\( r \) matroid.

(A) Show that if \( r^*(M) \leq r \) and every hyperplane is a circuit, then every circuit is a hyperplane.

(B) Use duality to show that if \( r^*(M) \geq r \) and every circuit is a hyperplane, then every hyperplane is a circuit.

(C) Give an example to show that (9A) and (9B) can fail if the condition on \( r^*(M) \) are dropped.

(3.2.20) Exercise: Let \( C_1^*, \ldots, C_r^* \) be cocircuits of a rank-\( r \) matroid \( M \). Prove the following are equivalent.

(A) \( M \) has a cobasis \( B^* \) such that \( C_1^*, \ldots, C_r^* \) are all of the fundamental cocircuits with respect to \( B^* \).

(B) For all \( j \in \{1, 2, \ldots, r\} \), \( C_j^* \not\subseteq \bigcup_{i \neq j} C_i^* \).

(3.2.21) Exercise: A cyclic flat of a matroid \( M \) is a flat of \( M \) that is the union of a (possibly empty) set of circuits. Show that

(A) \( X \) is a flat of \( M \) iff \( E(M) - X \) is the union of a (possibly empty) set of cocircuits of \( M \). Deduce that \( F \) is a cyclic flat of \( M \) iff \( E(M) - F \) is a cyclic flat of \( M^* \).

(B) A flat \( F \) of \( M \) is cyclic iff \( M\mid F \) has no coloops.

(C) A matroid is uniquely determined by a list of its cyclic flats and their ranks.

(3.2.22) Exercise: Find all connected graphs whose cycle matroid have a circuit hyperplane. How about dropping the connectedness condition?

(3.2.23) Exercise: Let \( \mathcal{A} \) and \( \mathcal{D} \) be clutters on a set \( S \). Prove that the following are equivalent:

(A) \( \mathcal{D} = b(\mathcal{A}) \).

(B) For all \( A \in \mathcal{A} \) and all \( D \in \mathcal{D} \), \( A \cap D \neq \emptyset \). Moreover, if \( d \in D \), then \( \mathcal{A} \) has an element \( A_d \) such that \( D \cap A_d = \{d\} \).

(C) For all \( X \subseteq S \), \( X \) contains a member of \( \mathcal{D} \) iff \( S - X \) does not contain a member of
(3.2.24) Exercise: For $i = 1, 2$, let $A_i$ be $m_i \times n$ over a field $F$. Suppose that every row of $A_2$ is a linear combination of rows of $A_1$. Prove that $M \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = M[A_1]$.

(3.2.25) Exercise: Let $A$ be a non-zero matrix over a field $F$ and suppose $M[A]$ has no coloops. Show that no row of $A$ contains exactly one non-zero entry.

(3.2.26) Exercise: Show that in a binary matroid, a circuit and a cocircuit cannot have an odd number of common elements.

(3.2.27) Prove that $U_{n,n+2}$ is $F$-representable if and only if $|F| \geq n + 1$.

(3.2.28) Let $M_1$ and $M_2$ be matroids on set $E$ and $cl_1$ and $cl_2$ be their closure operators. Prove $M_1 = M_2^*$ iff, for every partition ($\{e\}, X, Y$) of $E$, the element $e$ is in exactly one of $cl_1(X)$ and $cl_2(Y)$.

3. Minors

(3.3.1) Example: Let $A$ be a matrix over a field $F$, and let $X \subseteq E = E(M[A])$. Then $M[A] - X \cong M[A - X]$.

(3.3.2) Example: Let $G$ be a graph and let $X \subseteq E(G)$. Denote $G - X$ to be the subgraph of $G$ obtained by deleting edges in $X$. Then $M(G - X) \cong M(G) - X$.

(3.3.3) Example: Let $M = U_{r,n}$ and $T \subseteq E(M)$ with $|T| = k$. Then $M - T = U_{r',n-k}$, where $r' = \max\{r, n - k\}$.

(3.3.4) Let $M$ be a matroid and let $X \subseteq E(M)$ be a subset. The contraction of $X$ from $M$, is given by $M/X = (M^* - X)^*$.
(3.3.5) Let \( M = (E, \mathcal{I}) \) be a matroid. Let \( R \subseteq E \) and \( B_R \in \mathcal{B}(M|R) \). Let
\[
\mathcal{I}' = \{ I \subseteq E - R : I \cup B_R \in \mathcal{I} \}.
\]
Then \( \mathcal{I}' \) satisfies (I1)-(I3). Moreover, \( \mathcal{I}(M/R) = \mathcal{I}' \).

**Proof:** Verify (I1)-(I3).

(3.3.6) Example: Let \( G \) be a graph and \( X \subseteq E(G) \). Denote \( G/X \) to be the graph obtained from \( G \) by identifying the two ends of each edge in \( X \) and then deleting the edges in \( X \). Then \( M(G/X) \cong M(G)/X \).

(3.3.7) Example: Let \( T \) be a \( t \) subset of \( E = E(U_{m,n}) \). Then
\[
U_{m,n} - T \cong \begin{cases} 
U_{n-t,n-t} & \text{if } n \geq t \geq n - m \\
U_{m,n} - t & \text{if } t < n - m 
\end{cases}
\]
\[
U_{m,n}/T \cong \begin{cases} 
U_{0,n-t} & \text{if } n \geq t \geq m \\
U_{m-t,n-t} & \text{if } t < m 
\end{cases}
\]

(3.3.8) Let \( r, r_{M/R}, r_{M-R} \) denote the rank functions of \( M, M/R \) and \( M - R \), respectively. Then
\[
r_{M/R}(X) = r(X \cup R) - r(R), \text{ for all } X \subseteq E - R,
\]
and
\[
r_{M-R}(X) = r(X), \text{ for all } X \subseteq R.
\]

**Proof:** Apply (3.3.5).

(3.3.9) Let \( T, T' \) be disjoint subsets of \( E \) and let \( M = (E, \mathcal{I}) \) be a matroid. Each of the following holds:
(i) \( (M - T) - T' = M - (T \cup T') = (M - T') - T \).
(ii) \( (M/T)/T' = M/(T \cup T') = (M/T')/T \).
(iii) \( (M - T)/T' = (M/T') - T \).
(iv) \( M - T = (M^*/T)^* \).
(v) \( (M/T)^* = M^* - T \).
(vi) \( M^*/T = (M - T)^* \).

(Any matroid \( N \) with the form \((M - T)/T'\) is called a minor of \( M \). )

**Proof:** Verify that both sides have the same independent sets (or same bases, etc).

(3.3.10) Let \( N \) be a minor of \( M \), then

\[ N = (M - X)/Y \text{ if and only if } N^* = (M^*/X) - Y. \]

**Proof:** Suppose that \( N = (M - X)/Y \). Then apply (3.3.9) to get

\[ N^* = ((M - X)/Y)^* = (M - X)^* - Y = M^*/X - Y. \]

Take the dual both sides to complete the proof.

(3.3.11) Exercise: Let \( A \) be a \( m \times n \) matrix over a field \( F \) with \( n \geq m \geq 2 \) and let \( e \) be a (labelled) column of \( A \). Let \( A - e \) denote the submatrix of \( A \) by deleting column \( e \). Then \( M[A] - e \cong M[A - e] \).

(3.3.12) Let \( A = \begin{bmatrix} I_r & D \end{bmatrix} \) and let \( e \) denote the first column of \( A \) and let \( A/e \) denote the matrix obtained from \( A \) by deleting the first row and the first column of \( A \). Then \( M[A]/e \cong M[A/e] \).

(3.3.13) If \( M \) is an \( F \)-representable matroid, then every minor of \( M \) is \( F \)-representable.

**Proof:** This follows from (3.3.11) and (3.3.12).

(3.3.14) Exercise: If \( M \) is a graphic matroid, then every minor of \( M \) is graphic.

(3.3.15) Exercise: Let \( T \subset E(M) \). Then

\[ \mathcal{I}(M/T) = \{ I \subseteq E - T : \exists B \in \mathcal{B}(M|T) \text{ such that } I \cup B \in \mathcal{I}(M) \}. \]

(3.3.16) Exercise: Let \( B_T \in \mathcal{B}(M|T) \). Then

\[ \mathcal{B}(M/T) = \{ B' \subseteq E - T : B' \cup B_T \in \mathcal{B}(M) \} \]
\[ = \{ B' \subseteq E - T : \exists B \in \mathcal{B}(M|T) \text{ such that } I \cup B \in \mathcal{I}(M) \}. \]
(3.3.17) Exercise: A subset $X \in \mathcal{C}(M/T)$ iff $X$ satisfies each of the following:

(i) $X \neq \emptyset$.

(ii) $X = C - T$, for some $C \in \mathcal{C}(M)$.

(iii) No proper nonempty subset of $X$ satisfies (ii).

(3.3.18) Exercise: For all $X \subseteq E - T$,

$$cl_{M/T}(X) = cl_{M}(X \cup T) - T.$$ 

(3.3.19) Exercise: Let $T \subseteq E = E(M)$, each holds.

(i) $\mathcal{I}(M - T) = \{I \subseteq E - T : I \in \mathcal{I}(M)\}$.

(ii) $\mathcal{C}(M - T) = \{C \subseteq E - T : C \in \mathcal{C}(M)\}$.

(iii) $cl_{M-T}(X) = cl_{M}(X) - T$.

(3.3.20) Exercise: $\mathcal{I}(M/T) \subseteq \mathcal{I}(M - T)$, and equality holds if and only if

$$r(T) + r(E - T) = r(M).$$

(3.3.21) Exercise: Suppose that, for all elements $f$ of a loopless matroid $M$, $r(M - f) = r(M)$, but that, for some elements $e$ and $g$, $r(M - \{e, g\}) = r(M \setminus e/g)$. Show that $\{e, g\}$ is a cocircuit of $M$.

(3.3.22) Exercise: Let $M$ be a matroid and $T$ be a subset of $E = E(M)$. Show that

(A) $M/T$ has no loops iff $T$ is a flat of $M$.

(B) $cl(T) = T \cup \{e \in E - T : e$ is a loop of $M/T\}$.

(C) $cl^*(T) = T \cup \{e \in E - T : e$ is a coloop of $M/T\}$.

(3.3.23) Exercise: Let $M$ be a matroid, $T \subseteq E = E(M)$, and $B_T \in \mathcal{B}(M|T)$. Prove directly that $M/T$ is a matroid by showing that

$$\mathcal{B}_T = \{B' \subseteq E - T : B' \cup B_T \in \mathcal{B}(M)\}$$

is the set of a matroid on $E - T$. 

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(3.3.24) Exercise: Let \( T \) be a subset of \( E = E(M) \). Prove the following are equivalent.

(A) \( M - T = M/T \).
(B) \( r(M - T) \leq r(M/T) \).
(C) \( M \) has no circuits meeting both \( T \) and \( E - T \).

4. Elementary Constructions

(3.4.1) So far, we have discussed these different ways to get new matroids from old ones.

- Minor: Restriction or deletion (E1.1.8), and Contraction (2.2.2)
- Union (E1.1.4)
- Truncation (E1.2.20)
- Elongation (E1.2.21)
- Relaxation (2.1.4)

(3.4.2) Exercise: For \( i = 1, 2 \), let \( M_i = (E_i, I_i) \) be a matroid on disjoint sets \( E_1 \) and \( E_2 \), let \( E = E_1 \cup E_2 \), and let \( I = \{X = X_1 \cup X_2 : X_i \in I_i\} \).

(i) Then \( I \) is the collection of independent sets of a matroid \( M \) on \( E \). (This matroid \( M = (E, I) \) is called the direct sum of \( M_1 \) and \( M_2 \), usually denoted by \( M = M_1 \oplus M_2 \).)

(ii) Find examples of: direct sum of two graphic matroids, direct sum of two matric matroids.

(iii) Determine the circuits, cocircuits, bases, cobases, rank and corank of \( M_1 \oplus M_2 \) in terms of those of \( M_1 \) and \( M_2 \).

(iv) Define the direct sum of \( n \) matroids, where \( n \geq 2 \) is an integer.

(v) \( (M_1 \oplus M_2)^* = M_1^* \oplus M_2^* \).

(vi) \( M_1 \oplus M_2 \) is \( F \)-representable iff each \( M_i \) is \( F \)-representable. Thus \( F_7 \oplus F_7^* \) is not representable over any field.

(3.4.3) Exercise: For \( i = 1, 2 \), let \( M_i = (E_i, I_i) \) be a matroid on disjoint sets \( E_1 \) and \( E_2 \), and let \( E = E_1 \cup E_2 \). Let \( p \notin E \) be an element. Take a \( p_i \in E_i \) such that \( p_i \) is neither a loop nor a coloop of \( E_i \), (1 \( \leq i \leq 2 \)). (A loop of \( M \) is an element \( e \in E \) such that \( r_M(\{e\}) = 0 \). A coloop of \( M \) is a loop of \( M^* \).)
(i) Let
\[ C_S = C(M_1 - p_1) \cup C(M_2 - p_2) \]
\[ \cup \{(C_1 - p_1) \cup (C_2 - p_2) \cup p : p_i \in C_i \in C(M_i), 1 \leq i \leq 2\}. \]
Show that \( C_S \) is the collection of circuits of a matroid \( M \) on \( E \). This matroid \( M \) is called the \textbf{series connection} of \( M_1 \) and \( M_2 \) with respect to the basepoints \( p_1 \) and \( p_2 \), and is denoted by \( S(M_1; p_1, M_2; p_2) \).

(ii) Let
\[ C_P = C(M_1 - p_1) \cup \{(C_1 - p_1) \cup p : p_1 \in C_1 \in C(M_1)\} \]
\[ \cup C(M_2 - p_2) \cup \{(C_2 - p_2) \cup p : p_2 \in C_2 \in C(M_2)\} \]
\[ \cup \{(C_1 - p_1) \cup (C_2 - p_2) p_i \in C_i \in C(M_i), 1 \leq i \leq 2\}. \]
Show that \( C_P \) is the collection of circuits of a matroid \( M \) on \( E \). This matroid \( M \) is called the \textbf{parallel connection} of \( M_1 \) and \( M_2 \) with respect to the basepoints \( p_1 \) and \( p_2 \), and is denoted by \( P(M_1; p_1, M_2; p_2) \).

(3.4.4) Exercise: Determine \( P(M_1; p_1, M_2; p_2) \) and \( S(M_1; p_1, M_2; p_2) \) when \( M_1 \cong M_2 = U_{1,1} \).

(3.4.5) Exercise: Find examples to describe the series and parallel connections of two graphic matroids, and of two \( F \)-representable matroids.

(3.4.6) \textbf{Serial-Parallel Networks} Let \( M, N \) be matroids. If \( \{f, e\} \in C(M) \) and if \( M - f = N \), then \( M \) is a \textbf{parallel extension} of \( N \) and \( N \) is a \textbf{parallel deletion} of \( M \). In this case, \( f \) is \textbf{in parallel with} \( e \).

If \( \{f, e\} \in C(M^*) \) and if \( M/f = N \), then \( M \) is a \textbf{serial extension} of \( N \) and \( N \) is a \textbf{serial contraction} of \( M \). In this case, \( f \) is \textbf{in series with} \( e \).

A matroid \( N \) is a \textbf{series minor} of a matroid \( M \) if \( N \) is obtained from \( M \) by a sequence of deletions and serial contractions. A matroid \( N \) is a \textbf{parallel minor} of a matroid \( M \) if \( N \) is obtained from \( M \) by a sequence of contractions and parallel deletions.

(3.4.7) Exercise: \( M \) is a parallel extension of \( N \) iff \( M^* \) is a series extension of \( N^* \); \( N \) is a series minor of \( M \) if and only if \( N^* \) is a parallel minor of \( M^* \).
4. Matroid Intersection and Partition

1. Maximum Common Independent Set Problem

In this section, we always assume that $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ are two matroids on the common ground set $E$, and $r_1 = r_{M_1}$ and $r_2 = r_{M_2}$ are the rank functions of $M_1$ and $M_2$, respectively.

(4.1.1) Given two matroid $M_1$ and $M_2$ with $E(M_1) = E(M_2) = E$, the Maximum Common Independent Set Problem seeks a subset $J^* \in I(M_1) \cap I(M_2)$ such that

$$|J^*| = \max\{|I| : J \in I(M_1) \cap I(M_2)|.$$

(4.1.1A) More generally, we can even consider two weighted matroids $M_1$ and $M_2$ with $E(M_1) = E(M_2)$, such that $c_1$ is the weight function of $M_1$ and $c_2$ is the weight function of $M_2$. The Weighted Matroid Maximum Common Independent Set Problem seeks a subset $J^* \in I(M_1) \cap I(M_2)$ such that

$$c_1(J^*) + c_2(J^*) = \max\{c_1(I) + c_2(I) : J \in I(M_1) \cap I(M_2)|.$$ Unless otherwise stated, we will only consider the unweighed case in this lecture note.

(4.1.2) Example: (Bipartite Matching) Let $G$ be a bipartite graph with vertex bipartition $S$ and $T$, and with $E = E(G)$. Let $M_1 = (E, I_1)$ be a matroid such that an edge subset $J \in I_1$ if and only if no two edges in $J$ share a common vertex in $S$; and $M_2 = (E, I_2)$ be a matroid such that $J \in I_2$ if and only if no two edges in $J$ share a common vertex in $T$. Therefore, an edge subset $J$ is in $I_1 \cap I_2$ if and only if $J$ is a matching of $G$, and an optimal solution of Problem (4.1.1) in this case corresponds to a maximum matching.

(4.1.3) Example (Optimal Branching, [4] and [?]) Let $G$ be a graph with a distinguished vertex $v'$. Let $D = D(G)$ be an orientation of $G$. For each arc (directed edge) $e$ oriented from a vertex $u$ (the tail of $e$) to a vertex $v$ (the head of $e$), we assume that $e$ is capable to transmit messages from $u$ to $v$. Moreover, the distinguished vertex $v'$ is a source (a vertex from which messages are coming out). We may also assume that there is a function
$c : E(D) \to \mathbb{R}$ such that for each arc $e$, $c(e)$ is the cost for transmitting a unit message through $e$.

The **Optimal Branching** problem seeks an arc subset $J$ such that messages from $v'$ can be sent to any other vertices, such that the cost on $J$ is minimized. (Thus $J$ is an arborescence rooted at $v'$ with minimum cost).

Let $E = E(G)$ as well as the set of all directed edges in $D(G)$. Define $M_1 = (E, \mathcal{I}_1)$ such that $M_1 = M(G)$ is the cycle matroid of the undirected graph $G$. To define another matroid $M_2$, we let

$$
V(G) = \{v_1, v_2, \ldots, v_n\} \text{ with } v' = v_1,
$$

and for each $i = 1, 2, \ldots, n$, let $A_i = E_D(v_i)$ denote the set of all arcs directed into $v_i$. (Thus $A_1, A_2, \ldots, A_n$ is a partition of $E$). Define $M_2 = (E, \mathcal{I}_2)$ such that

$$
\mathcal{I}_2 = \{J \subseteq E : |J \cap A_i| \leq 1, \quad 2 \leq i \leq n, \quad \text{and } |J \cap A_1| = 0\}.
$$

Define another weight function $c_1 : E \to \mathbb{R}$ by

$$
\forall e \in E, c_1(e) = 1 - c(e) + \sum_{x \in E} c(x).
$$

**Claim 1** $J$ is an arborescence of $D$ rooted at $v'$ if and only if $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|J| = n - 1$.

Let $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $|J| = n - 1$. Since $J \in \mathcal{I}_1$, $J$ induces a spanning tree in $G$. As

$$
n - 1 = |J| = \sum_{i=2}^{n} |J \cap A_i| \leq n - 1,
$$

we must have $|J \cap A_i| = 1$, for each $i = 2, 3, \ldots, n$. This implies that $J$ must be an arborescence rooted at $v' = v_1$.

Now suppose that $J$ is an arborescence of $D$ rooted at $v'$. Then $J$ is a spanning tree in $G$, and $|J \cap A_i| = 1$, for each $i = 2, 3, \ldots, n$. Therefore, $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $|J| = n - 1$.

**Claim 2** Any optimal solution of (4.1.1A) must be a maximum cardinality subset in $\mathcal{I}_1 \cap \mathcal{I}_2$.

Consider Problem (4.1.1A) with $c_2 \equiv 0$. Suppose that $J_1, J_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$ and such that $|J_1| > |J_2|$. Then

$$
\sum_{e \in J_1} c_1(e) - \sum_{e \in J_2} c_1(e) = \left(1 + \sum_{x \in E} c(x)\right)(|J_1| - |J_2|) - \sum_{e \in J_1} c(e) + \sum_{e \in J_2} c(e)
\geq \sum_{x \in E} c(x) + 1 - \sum_{x \in J_2} c(x) + 0 > 0.
$$
Thus an optimal solution of (4.1.1A) must be an arborescence $J^*$ rooted at $v'$ such that $\sum_{e \in J^*} c_1(e)$ is maximized (which is equivalent to $\sum_{e \in J^*} c(e)$ is minimized).

**Remark:** If we only consider the undirected graph, then we are looking for a minimum spanning tree, which does not require a solution of Problem (4.1.1).

(4.1.4) Let $J \in I(M_1) \cap I(M_2)$. Then each of the following holds.

(i) $\forall X \subseteq E, |J| \leq r_1(X) + r_2(E - X)$.

(ii) If for some $X \subseteq E$, such that $|J| = r_1(X) + r_2(E - X)$, then $J$ must be a solution of Problem (4.1.1).

**Proof:** (i) Since $J \in I(M_1) \cap I(M_2)$, by the definition of rank functions,

$$|J| = |J \cap X| + |J \cap (E - X)| \leq r_1(X) + r_2(E - X).$$

(ii) Let $J'$ be a solution of Problem (4.1.1). Suppose that for some $X \subseteq E$, we have $|J| = r_1(X) + r_2(E - X)$. Then since $J'$ is a solution of Problem (5.1.1) and by (i),

$$|J| \leq |J'| \leq r_1(X) + r_2(E - X) = |J|,$$

and so we must have $|J| = |J'|$, and so $J$ must also be a solution of Problem (4.1.1).

(4.1.5) Let $k$ be a nonnegative integer. If for any subset $X \subseteq E$,

$$k \leq r_1(X) + r_2(E - X),$$

then there exists a subset $J \subseteq E$ such that $J \in I_1 \cap I_2$.

**Proof:** We argue by induction on $|E|$. If $k = 0$ or $|E| \leq 1$, then the conclusion is trivial. Therefore, we assume $k \geq 1$ and $|E| \geq 2$.

**Case 1:** For any non-empty proper subset $X$, (that is $\emptyset \neq X \neq E$), we always have $k < r_1(X) + r_2(E - X)$.

Pick $e \in E$, let $M'_i = M_i - e$, $(1 \leq i \leq 2)$, and let $r_i$ denote the rank function of $M'_i$. Then for any subset $X' \subseteq E' = E - e$, let

$$X = \begin{cases} X' \cup e & \text{if } X' \neq E' \\ X' & \text{if } X' = E' \end{cases}.$$
As \(|E| \geq 2\), we have \(\emptyset \neq X \neq E\). By the assumption of Case 1,

\[ r'_1(X') + r'_2(E' - X') \geq r_1(X) + r_2(E - X) - 1 \geq k. \]

Therefore by induction on \(|E|\), \(\exists J' \in \mathcal{I}(M'_1) \cap \mathcal{I}(M'_2) \subseteq \mathcal{I}_1 \cap \mathcal{I}_2\) such that \(|J'| \geq k\).

**Case 2:** For some non-empty proper subset \(X\), \(k = r_1(X) + r_2(E - X)\).

Let \(M''_1 = M_1/X\) and \(M''_2 = M_2 - X\), and let \(r''_1, r''_2\) denote the rank functions of \(M''_1\) and \(M''_2\), respectively. Then for any subset \(T \subseteq E - X\),

\[ r''_1(T) + r''_2(T) = r_1(T \cap X) - r_1(X) + r_2(E - (X \cup T)) \geq k - r_1(X) = r_2(E - X). \]

Since \(E \neq X \neq \emptyset\), \(0 < |E - X| < |E|\). By induction, \(\exists J_2 \in \mathcal{I}(M_1/X) \cap \mathcal{I}(M_2 - X)\) such that \(|J_2| \geq r_2(E - X)\).

Let \(Y = E - X\). Then working with \(M_2/Y\) and \(M_1 - Y\), we can similarly find \(J_1 \in \mathcal{I}(M_1 - Y) \cap \mathcal{I}(M_2/Y) = \mathcal{I}(M_1/X) \cap \mathcal{I}(M_2/(E - X))\) such that \(|J_1| \geq r_1(X)\).

Let \(J = J_1 \cap J_2\). Then since \(J_2 \in \mathcal{I}(M_1/X)\), there exist a base \(B_1 \in \mathcal{B}(M_1/X)\) such that \(J_2 \subseteq B_1\). Since \(J_1 \in M_1/X\), by the definition of contraction, \(J_1 \cup B_1 \in \mathcal{I}(M_1)\). It follows that \(J = J_1 \cup J_2 \subseteq J_1 \cup B_1 \in \mathcal{I}(M_1)\), and so \(J_1 \in \mathcal{I}(M_1)\). Similarly, \(J \in \mathcal{I}(M_2)\). As \(|J| = |J_1| + |J_2| \geq r_1(X) + r_2(E - X) = k\), \(J\) is the desired subset.

(4.1.6) **(Edmonds’ Matroid Intersection Theorem, [5])** For two matroids \(M_1\) and \(M_2\) on the same set \(E\),

\[ \max\{|J| : J \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)\} = \min\{r_1(X) + r_2(E - X) : X \subseteq E\}. \]

**Proof:** Let \(l = \max\{|J| : J \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)\}\) and \(k = \min\{r_1(X) + r_2(E - X) : X \subseteq E\}\).

By (4.1.4), \(l \leq k\) and by (4.1.5), \(l \geq k\).

(4.1.7) For any subset \(J \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)\), define a digraph \(G(J)\) as follows: \(V(G(J)) = E \cup \{s, t\}\), where \(s\) and \(t\) are two new elements not in \(E\); and \(E(G(J))\) consists of the directed edges described below:

(i) for each \(e \in E - J\), if \(J \cup e \in \mathcal{I}(M_1)\), then \((e, t)\) is a directed edge in \(G(J)\);
(ii) for each \(e \in E - J\), if \(J \cup e \in \mathcal{I}(M_2)\), then \((s, e)\) is a directed edge in \(G(J)\);
(iii) For each \( e \in E - J \) and \( f \in J \), if \((J \cup e) - f \in \mathcal{I}(M_1)\), then \((e, f)\) is a directed edge in \( G(J) \);

(iv) For each \( e \in E - J \) and \( f \in J \), if \((J \cup e) - f \in \mathcal{I}(M_2)\), then \((f, e)\) is a directed edge in \( G(J) \).

(4.1.8) **(Alternating Path Theorem)** Fix a subset \( J \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2) \), and let \( G = G(J) \).

(i) If \( G \) has a chordless \((s, t)\)-dipath \( s, e_1, f_1, \cdots, e_k, f_k, e_{k+1}, t \), then \( J' = (J \cup \{e_1, e_2, \cdots, e_{k+1}\}) - \{f_1, f_2, \cdots, f_k\} \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2) \).

(ii) If for some subset \( X \subseteq E \), \( G \) does not have a directed edge \((u, v)\) such that \( u \in X \cup \{s\} \) and \( v \in (E - X) \cup \{t\} \) (that is, \( G \) does not have any \((s, t)\)-dipath), then \(|J| = r_1(X) + r_2(E - X)\).

**Sketch of Proof:**

(4.1.9) **Matroid Intersection Algorithm (MIA).** Given matroids \( M_1 \) and \( M_2 \) on the same set \( E \), and a subroutine checking independency in \( M_1 \) and in \( M_2 \), whose running time would not exceed a polynomial of \(|E|\), MIA finds a solution of Problem (4.1.1).

**Initialization:** Set \( J := \emptyset \).

**Iteration:**

Step 1 Find a chordless \((s, t)\)-dipath \( s, e_1, f_1, \cdots, e_k, f_k, e_{k+1}, t \) in \( G(J) \).

If none exists, STOP (the current value of \( J \) is a solution).

Step 2 Augment the current value of \( J \)

by setting \( J := J \cup \{e_1, e_2, \cdots, e_{k+1}\} \) — \( \{f_1, f_2, \cdots, f_k\} \).

Back to Step 1.

(4.1.10) The validity of Algorithm (4.1.9) follows from (4.1.8). We consider the complexity of Algorithm (4.1.9). Let \( m = |E| \). Then Algorithm (4.1.9) would stop after most \( m \) iterations. In each iteration, it takes \( O(m^2) \) times of checking independency to construct the graph \( G(J) \), and it also takes a polynomial time to find a chordless \((s, t)\)-dipath in \( G(J) \), or to conclude that \( G(J) \) does not have such a path. Therefore, MIA is a polynomial algorithm.

(4.1.11) **Exercise:** Let \( M_1 \) and \( M_2 \) be two matroids on the set \( E \). Show that \( I \) is a maximum common independent set in \( \mathcal{I}(M_1) \cap \mathcal{I}(M_2) \) if and only if there exist \( I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2) \) such that \( I = I_1 \cup I_2, I_1 \cap I_2 = \emptyset \) and \( cl_{M_1}(I_1) \cup cl_{M_2}(I_2) = E \).
Exercise: Let $M_1, M_2, \cdots, M_k$ be matroids on mutually disjoint sets $E_1, E_2, \cdots, E_k$, respectively. Let $S = E_1 \cup E_2 \cup \cdots \cup E_k$.

(i) Define $I'$ by $I \in I'$ if and only if for each $i$, $(1 \leq i \leq k)$, $I \cap E_i \in I(M_i)$. Then $I'$ satisfies the independence axioms and so is the collection of independent sets of a matroid on $S$.

(ii) Suppose, in addition, that for a set $E = \{e_1, e_2, \cdots, e_m\}$, there exist bijections $\pi_j : E \mapsto E_j$ with $\pi_i(e_j) = e_{ij}$, $1 \leq i \leq k$ and $1 \leq j \leq m$. Let $A_j = \{e_{1j}, e_{2j}, \cdots, e_{mj}\}$. Define $I''$ by $J \in I''$ if and only if for each $j$, $|J \cap A_j| \leq 1$. Then $I''$ satisfies the independence axioms and so is the collection of independent sets of a matroid on $S$.

2. Matroid Partitions and Matroid Unions

Let $M_1, M_2, \cdots, M_k$ be matroids on the set $E$. For a subset $J \subseteq E$, if there exist subsets $J_1, J_2, \cdots, J_k$ such that

(i) $J = \bigcup_{i=1}^k J_i$,
(ii) $J_i \cap J_j = \emptyset$, and
(iii) for each $i$, $(1 \leq i \leq k)$, $J_i \in I(M_i)$,

then $J$ is a partitionable subset (with respect to the matroids $M_1, M_2, \cdots, M_k$). The Matroid Partition Problem seeks a partitionable subset $J$ with maximum cardinality (with respect to the matroids $M_1, M_2, \cdots, M_k$), and such a $J$ is called a maximum partitionable subset.

Let $M_1, M_2, \cdots, M_k$ be matroids on the set $E$, and let $I$ be the set of all partitionable subsets. Then $I$ satisfies (I1), (I2) and (I3)” (see (1.1.9)), and so there is a matroid $\vee_{i=1}^k M_i$ such that $I(\vee_{i=1}^k M_i) = I$. (The matroid $\vee_{i=1}^k M_i$ is called the union of $M_1, M_2, \cdots, M_k$.)

Proof: It suffices to verify that $I$ satisfies (I3)”.

We argue by contradiction and assume that a counterexample exists such that $|E|$ is minimized.

If (I3)” has a violation in a subset $X$, then we consider the restrictions $M_i|X$ instead. Therefore, it suffices to show that all maximal partitionable subsets have the same cardinality.

By contradiction, we assume that there exists a maximal partitionable subset $J$, and
another partitionable subset $J'$ such that $|J'| > |J|$.

Since $J, J'$ are partitionable, we may assume that

$$J = F_1 \cup F_2 \cup \cdots \cup F_k, \quad J' = F'_1 \cup F'_2 \cup \cdots \cup F'_k,$$

such that $F_i, F'_i \in \mathcal{I}(M_i), (1 \leq i \leq k)$.

Since $|J'| > |J|$, we may assume that $|F'_1| > |F_1|$. Then by (13), $\exists e \in F'_1 - F_1$ such that $F_1 \cup e \in \mathcal{I}(M_1)$. Since $J$ is maximal, $e \in J$ and so by exchange elements, we may assume that $e \in F_1 \cap F'_1$.

Let $M'_1 = M_1/e$ and $M_i = M_i - e, (2 \leq i \leq k)$. As $J - e = (F_1 - e) \cup F_2 \cup \cdots \cup F_k$ and $J' - e = (F'_1 - e) \cup F'_2 \cup \cdots \cup F'_k$, we conclude that $F - e$ and $F' - e$ are partitionable with respect to the matroids $M'_1, M'_2, \cdots, M'_k$.

Since $|J - e| < |J' - e|$, by the minimality of $E$, $J - e$ is not a maximal partitionable with respect to the matroids $M'_1, M'_2, \cdots, M'_k$. Therefore, $\exists f \in (E - e) - J$ such that $(J - e) \cup f$ is partitionable with respect to the matroids $M'_1, M'_2, \cdots, M'_k$. Let $F''_1, F''_2, \cdots, F''_k$ be the corresponding partition of $(J - e) \cup f$ such that $F''_i \in \mathcal{I}(M'_i)$. Then $F''_1 \cup e, F''_2, \cdots, F''_k$ is a partition of $J \cup f$, contrary to the assumption that $J$ is a maximal partitionable subset.

(4.2.3) If there is a polynomial algorithm for determining independence in matroids $M_1, M_2, \cdots, M_k$ (on the same set $E$), then there is a polynomial algorithm checking if a given subset is partitionable.

**Proof:** Let $E_1, E_2, \cdots, E_k$ be disjoint copies of $E$. Let $M'$ and $M''$ be the matroids on $S = E_1 \cup E_2 \cup \cdots \cup E_k$ as defined in Exercise (4.1.12)(i) and (ii), respectively.

Then the set $E$ is partitionable with respect to $M_1, M_2, \cdots, M_k$ if and only if $M'$ and $M''$ have a common independent set of cardinality $|E|$.

(4.2.4) Exercise: Show that $E$ is partitionable with respect to $M_1, M_2, \cdots, M_k$ if and only if $M'$ and $M''$ have a common independent set of cardinality $|E|$; and complete the proof of (4.2.3).

(4.2.5) *(The rank function of the union)* Let $M_1, M_2, \cdots, M_k$ be matroids on the set $E$, and let $r_i$ denote the rank function of $M_i$. Then

$$\max\{|J| : J \text{ is partitionable}\} = \min\{|E - T| + \sum_{i=1}^{k} r_i(T) : T \subseteq E\}.$$
Proof: Let \( j^* = \max\{|J| : J \text{ is partitionable}\} \) and \( t_* = \min\{|E - T| + \sum_{i=1}^{k} r_i(T) : T \subseteq E\} \).

Let \( J \) be a partitionable subset and \( F_1, F_2, \ldots, F_k \) be a corresponding partition, and let \( T \subseteq E \) be a subset. Then
\[
|J| = |J \cap (E - T)| + |J \cap T| \leq |E - T| + \sum_{i=1}^{k} |F_i \cap T| \leq |E - T| + \sum_{i=1}^{k} r_i(T).
\]

Hence \( j^* \leq t_* \).

Let \( M' \) and \( M'' \) be the matroids in (4.1.12)(i) and (ii), and let \( r', r'' \) be the rank functions of \( M' \) and \( M'' \), respectively. By Edmonds Matroid Intersection Theorem (4.1.6), there exist subsets \( J', T' \subseteq S \) such that
\[
r'(T') + r''\left( \bigcup_{i=1}^{k} E_i - T' \right) = |J'|.
\]

We shall use the notation in (4.1.12), and let \( A_j = \{e_{1j}, e_{2j}, \ldots, e_{mj}\} \). Suppose that for some \( j \), \( A_j - T' \neq \emptyset \). For each such \( j \), we remove the elements in \( A_j \cap T' \) from \( T' \) so that for such \( j \), \( A_j \cap T' = \emptyset \). (One can view this as introducing \( T'' \) from \( T' \) by removing all the elements in \( A_j \cap T' \) whenever \( A_j - T' \neq \emptyset \), then redefine \( T' \) as \( T'' \)). Note that such a transfer of elements does not increase \( r''\left( \bigcup_{i=1}^{k} E_i - T' \right) \). Then let
\[
T = \{e_j : A_j \subseteq T'\}, \quad \text{and} \quad J = \{e_j : A_j \cap J' \neq \emptyset\}.
\]

Then
\[
\begin{align*}
j^* & \geq |E - T| + \sum_{i=1}^{k} r_i(T) = r''\left( \bigcup_{i=1}^{k} E_i - T' \right) + r'(T') \\
& \leq |J'| = |J| \geq t_*.\end{align*}
\]

(4.2.6) **Edmonds Matroid Partition Theorem** ([2]) Let \( M_1, M_2, \ldots, M_k \) be matroids defined on a set \( E \), and \( r_i \) denote the rank function of \( M_i \).

(i) If \( J \subseteq E \) is a partitionable subset, the for any subset \( X \subseteq E \),
\[
|J| \leq |E - X| + \sum_{i=1}^{k} r_i(X).
\]
(ii) If \( J \subseteq E \) is a maximum partitionable subset, then \( |J| = |E - X| + \sum_{i=1}^{k} r_i(X) \).

**Proof:** Already done in the above. Left as an exercise.

(4.2.7) **Exercise:** If \( E \) is partitionable with respect to matroids \( M_1, M_2, \ldots, M_k \), and if \( r_i \) is the rank function of \( M_i \), then

\[
\forall X \subseteq E, \sum_{i=1}^{k} r_i(X) \geq |X|.
\]

(4.2.8) **Matroid Partition Algorithm** When (4.2.7) is satisfied, the following algorithm finds subsets \( I_1, I_2, \ldots, I_k \) such that \( E = \bigcup_{i=1}^{k} I_i \) and such that for each \( i \), \( I_i \in \mathcal{I}(M_i) \).

**Notation:** For \( i = 1, 2, \ldots, k \), \( r_i \) and \( cl_i \) denote the rank and the closure operator of \( M_i \), respectively.

**Initialization:** Set \( I_i := \emptyset, \forall i = 1, 2, \ldots, k \).

\( S_0 := E, j := 1, U := E - \bigcup_{i=1}^{k} I_i \).

**Iteration Step**

1. Pick \( e \in U \).
2. Let \( D := \{i|I_i \cap S_{j-1}| < r_i(S_{j-1}), 1 \leq i \leq k\} \).

Choose \( i \in D \) such that \( i \) is the smallest index in \( D \).
3. Let \( S_j := S_{j-1} \cap cl_i(I_i \cap S_{j-1}) \). Set \( e(j) := i \).
4. If \( e \in S_j \), then update \( j := j + 1 \), GOTO Step 2.
5. (It must be \( e \notin S_j \)). If \( I_{e(j)} \cup e \in \mathcal{I}(M_{e(j)}) \), then update \( I_{e(j)} := I_{e(j)} \cup e \), GOTO Step 4.

If \( I_{e(j)} \cup e \) contains a circuit \( C \), then \( C - S_{j-1} \neq \emptyset \) (as otherwise \( C - e \subseteq I_{e(j)} \) and \( C \subseteq S_{j-1} \), and so \( e \in cl_{e(j)}(I_{e(j)} \cap S_{j-1}) \), contrary to \( e \notin S_j \)).

Pick \( e' \in C - S_{j-1} \).
6. Let \( m := \min\{q : e' \notin S_q, 1 \leq q \leq j - 1\} \). Update \( I_{e(j)} := (I_{e(j)} \cup e) - e' \), and set \( e := e' \). GOTO Step 5.
7. If \( U - e = \emptyset \), STOP (Now the current values of \( I_1, I_2, \ldots, I_k \) are the desired output); OTHERWISE set \( U := U - e \), GOTO Step 1.

(4.2.9) **Exercise:** Show the validity of Algorithm (4.2.8).
5. Matroid Connectivity and Decomposition

1. Connected Matroids

(5.1.1) Define a relation $\sim$ on $E(M)$ as follows: $\forall e, e' \in E(M)$, $e \sim e'$ if $e = e'$ or $\exists C \in \mathcal{C}(M)$ such that $e, e' \in C$. Then $\sim$ is an equivalence relation. (Each equivalence class, as a restriction of $M$, is a component of $M$. $M$ is connected if $M$ has only one component, and $M$ is disconnected if $M$ has more than one components.)

(5.1.2) Let $G$ be a connected graph. Then $M(G)$ is connected if and only if either $|V(G)| \leq 2$, or $G$ is 2-connected.

(5.1.3) $U_{r,n}$ is connected iff $1 \leq r \leq n - 1$.

(5.1.4) Let $M = (E, \mathcal{I})$ be a matroid on $E$ with rank function $r$. Define the connectivity function of $M$ by

$$h(X) = r(X) + r(E - X) - r(E), \text{ for any } X \subseteq E.$$ 

Then

(i) $h(X) = h(E - X) = r_M(X) + r_{M^*}(X) - |X| \geq 0$.

(ii) $M$ and $M^*$ have the same connectivity function.

(iii) Both $M_1 \oplus M_2$ and $M_1^* \oplus M_2^*$ have the same connectivity function.

(5.1.5) Let $k > 0$ be an integer. Let $M = (E, \mathcal{I})$ be a matroid. A subset $X \subseteq E$ is a separator of $M$ if $h(X) = 0$. Then $M$ is connected iff $M$ has no separators.

(5.1.6) (Decomposition into 1-sums) Let $M$ be a matroid. If the $\sim$ equivalence classes are $E_1, E_2, \ldots, E_c$, then

$$M = (M|E_1) \oplus (M|E_2) \oplus \cdots \oplus (M|E_c).$$

(5.1.7) Exercise: Show that a matroid $M$ is connected iff, for every pair of distinct elements of $E(M)$, there is a hyperplane avoiding both.
(5.1.8) Exercise: Show that every component of a loopless matroid is closed.

(5.1.9) Exercise: If $X$ is a subset of a matroid $M$ and $M|X$ is connected, find necessary and sufficient conditions for $M|(cl(X))$ to be connected.

(5.1.10) Exercise: Let $X$ and $Y$ be subsets of a matroid $M$ such that both $M|cl(X)$ and $M|cl(Y)$ are connected and $X \cap Y \neq \emptyset$. Show that $M|(cl(X \cup Y))$ is connected.

(5.1.11) Exercise: Let $A$ be an $r \times n$ matrix over the field $F$. Prove that $M[A]$ is connected iff there is no partition of the set of columns of $A$ into non-empty sets $X_1$ and $X_2$ such that $\langle X_1 \rangle \cap \langle X_2 \rangle = \{0\}$.

(5.1.12) Exercise: Let $T \subseteq E = E(M)$. Prove that TFAE:

(A) $T$ is a separator of $M$.
(B) $M^* - T = M^*/T$.
(C) Every hyperplane of $M$ contains $T$ or $E - T$.
(D) $T$ is a separator of $M^*$.

(5.1.13) Exercise: Show that if $\{e, f\} \in C(M) \cap C(M^*)$, then $\{e, f\}$ is a component of $M$.

(5.1.14) Exercise: Let $M$ be a matroid and let $r^* = r_{M^*}$. If $X \subseteq E = E(M)$, show that $r^*(X) = r((M|X)^*)$ iff $X$ is a separator of $M$.

(5.1.15) Exercise: Suppose that $A_1$ and $A_2$ are $F$-representations of the matroids $M_1$ and $M_2$. Show that \[
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix}
\] is an $F$-representation of $M_1 \oplus M_2$.

(5.1.16) Exercise: Without using matroid theory, prove directly that in a graph $G$, if every pair of distinct edges lie in a circuit if and only if every pair of distinct edges lie in a bond.
2. Tutte-Connectivity

(5.2.1) A \textbf{k-separation} of \( M \) is a partition \((X, Y)\) of \( E \) such that \( E = X \cup Y, X \cap Y = \emptyset, \min\{|X|, |Y|\} \geq k \) and \( h(X) \leq k - 1 \). The \( k \)-separation \((X, Y)\) is \textbf{minimal} if \( \min\{|X|, |Y|\} = k \), and is \textbf{exact} if \( h(X) = k - 1 \). (Thus \( X \) is a separator iff \((X, E - X)\) is a 1-separation.)

(5.2.2) Let \( G \) be a 2-connected graph with \(|E(G)| \geq 2k\) and let \( C \) be a \( k \)-circuit of \( G \). Then \((C, E - C)\) is an exact \( k \)-separation.

(5.2.3) If \( M \) has a \( k \)-separation, then \( M \) is \textbf{k-separated}. The \textbf{Tutte-connectivity} (or just \textbf{connectivity} for short) of \( M \), is

\[
\lambda(M) = \begin{cases} 
\min\{j : M \text{ is } j\text{-separated}\}, & \text{if } G \text{ is } k\text{-separated for some } k, \\
\infty & \text{otherwise}.
\end{cases}
\]

If \( \lambda(M) \geq k \), then \( M \) is Tutte \( k \)-connected, (or just \( k \)-connected).

(5.2.4) Let \( M = (E, \mathcal{I}) \) be a matroid.

(i) \( \lambda(M) = \lambda(M^*) \).

(ii) If \( X \) is a separator of \( M \), then \( Y = E - X \) is also a separator of \( M \).

(iii) If \( M = M_1 \oplus M_2 \), where \( E_i = E(M_i) \), then each of \( E_1 \) and \( E_2 \) is a separator of \( M \).

(\( M \) has a separator iff \( M \) has 1-separation).

(iv) \( M \) is 2-connected if and only if \( M \) is connected.

(v) (Exercise) The only connected matroid \( M \) with at most 3 elements are

\[ \{U_{0,0}, U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}\}. \]

(vi) (Exercise) the only 3-connected matroid \( M \) with at most 5 elements are

\[ \{U_{0,0}, U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}, U_{2,4}, U_{2,5}, U_{3,5}\}. \]

(5.2.5) Exercise: Let \( X \subset E(U_{r,n}) \) be a \( k' \)-subset. Find sufficient/necessary conditions such
that \((X, E - X)\) is a \(k\)-separation.

(5.2.6) Exercise: Let \(M\) be a matroid.

(i) If \(M\) is \(n\)-connected and if \(|E(M)| \geq 2(n - 1)\), then all circuits and cocircuits of \(M\) have at least \(n\) elements.

(ii) If \(M\) is \(n\)-connected and if \(|E(M)| \geq 2n - 1\), then \(M\) does not have an \(n\)-element subset that is both a circuit and a cocircuit.

(5.2.7) Exercise: Let \(M\) be a matroid.

(i) If \(\lambda(M) = \infty\), then \(M\) is a uniform matroid.

(ii) If \(M = U_{r,n}\), a uniform matroid, then

\[
\lambda(M) = \begin{cases} 
    r + 1 & \text{if } n \geq 2r + 2 \\
    n - r + 1 & \text{if } n \leq 2r - 2 \\
    \infty & \text{otherwise} 
\end{cases}
\]

(5.2.8) Exercise: Let \(M\) and \(N\) be matroids such that \(M - e = N\). Suppose that \(N\) is \(n\)-connected but \(M\) is not. Then one of the following must holds:

(i) \(e\) is a coloop of \(M\), or

(ii) \(M\) has a circuit that contains \(e\) and has fewer than \(n\) elements.

(Hint: Assume \(e\) is not a coloop. \(M\) has a \(k\)-separation \((X, Y)\) for some \(k < n\) with \(e \in X\). Use \(\lambda(N) \geq n\) to get \(|X| = k\) and \(r(X) = r(X - e)\).)

(5.2.9) Exercise: Let \(M(X, Y)\) be a \(k\)-separator of a matroid \(M\) and suppose that \(|X| = k\). Then \(X\) is either a coindependent circuit or an independent cocircuit.

(5.2.10) Exercise: \(X \subseteq E(M)\) is a flat if \(cl(X) = X\). Let \((X, Y)\) be a \(k\)-separator of a matroid \(M\) and suppose that \(|Y| \geq k + 1\). Then either \(X\) is both a flat and a coflat of \(M\), or, for some element \(e \in Y\), \((X \cup e, Y - e)\) is a \(k\)-separator of \(M\).

(5.2.11) Exercise: If \(e\) is an element of an \(n\)-connected matroid \(M\), and if \(|E(M)| \geq 2(n - 1)\), then both \(M - e\) and \(M/e\) are \((n - 1)\)-connected.

(5.2.12) Exercise: Let \(M'\) be obtained from \(M\) by relaxing a hyperplane-circuit. Show that \(\lambda(M') \geq \lambda(M)\).
(5.2.13) Exercise: Find $\lambda(F_7)$, $\lambda(F_7^{-})$, $\lambda(\text{AG}(3, 2))$.

(5.2.14) Exercise: Let $M$ be a matroid and let $(X_1, X_2)$ (with $X_2 = E - X_1$) be a partition of $M$ such that $\min\{|X_1|, |X_2|\} \geq 1$. Let $M_i = M - X_i$. Then the following are equivalent.

(i) $M = M_1 \oplus M_2$ (the direct sum of $M_1$ and $M_2$, see Exercise (1.1.10)).
(ii) $(X_1, X_2)$ is a 1-separation of $M$.

3. Regular and Binary Matroids

(5.3.1) Let $M$ be a matroid and $F$ be a field. If for some matrix $A$ over $F$, $M \cong M_F[A]$, then $M$ is an $F$-representable matroid, and $M_F[A]$ is a representation of $M$ over $F$. If $M$ is representable over $GF(2)$, the two element field, then $M$ is a binary matroid; if for any field $F$, $M$ is representable over $F$, then $M$ is a regular matroid.

(5.3.2) Some facts:
(i) Every regular matroid if binary.
(ii) $F_7$ is binary but $F_7^{-}$ is not regular.
(iii) Let $E = \{e_1, e_2, \cdots, e_m\}$ be a set. For each subset $X \subseteq E$, let

$$v_X(e_i) = \begin{cases} 1 & \text{if } e_i \in X \\ 0 & \text{if } e_i \notin X \end{cases}$$

denote the characteristic function of $X$, viewed as an $m$-dimensional vector in $V(m, 2)$, the $m$-dimensional vector space over $GF(2)$. Note that the bijection $X \leftrightarrow v_X$ gives a one-to-one correspondence between all subsets of $E$ and vectors in $V(m, 2)$.

Let $W$ be a subspace of $V(m, 2)$, and let $\mathcal{C}$ denote the minimal subsets of $E$ in $W$ under the correspondence above. Then $\mathcal{C}$ is the set of circuits of a matroid $M(W)$ on $E$. (Note that every subset in $W$ is a disjoint union of subsets in $\mathcal{C}$.)

(5.3.3) Let $N_1, N_2, \cdots, N_k$ be matroids. The collection of all matroids that do not have any of the $N_i$’s as a minor is denoted by $EX(N_1, N_2, \cdots, N_k)$. 

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(5.3.4) (Tutte [13]) A matroid $M$ is regular if and only if $M \in \text{EX}(U_{2,4}, F_7, F_7^*)$.

(5.3.5) (Tutte [14]) A matroid $M$ is graphic if and only if $M \in \text{EX}(U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_3,3))$.

(5.3.6) (Seymour [9], Tutte [13]) A matroid $M$ is binary if and only if $M \in \text{EX}(U_{2,4})$.

(5.3.7) Let $M$ be a matroid. A cycle of $M$ is a disjoint union of circuits of $M$. (In particular, the empty set, as it is the empty union of circuits, is a cycle). Let $C_0(M)$ denote the set of all cycles of $M$. Then $C(M) \cup \{\emptyset\} \subseteq C_0(M)$.

For two sets $X$ and $Y$, the symmetric difference of $X$ and $Y$ is $X \Delta Y = (X \cup Y) - (X \cap Y)$.

(5.3.8) Let $M$ be a matroid. The following are equivalent.

(i) $M$ is binary.

(ii) If $C_1, C_2 \in C(M)$ such that $C_1 \neq C_2$, then $\exists C_3 \in C(M)$ such that $C_3 \subseteq C_1 \Delta C_2$.

(iii) If $C \in C(M)$ and $C^* \in C(M^*)$, then $|C \cap C^*| \equiv 0 \pmod{2}$.

(iv) If $C_1, C_2, \ldots, C_n \in C(M)$, then either $\Delta_{i=1}^n C_i = \emptyset$, or for some $C \in C(M)$ such that $C \subseteq \Delta_{i=1}^n C_i$.

(v) If $C_1, C_2, \ldots, C_n \in C(M)$, then $\Delta_{i=1}^n C_i \in C_0(M)$.

4. Decomposition of Regular Matroids

(5.4.1) Let $M_1$ and $M_2$ be two binary matroids on $E_1 = E(M_1)$ and $E_2 = E(M_2)$, respectively. Suppose that $|E_1 \cap E_2| \leq 3$. Let $C_0 = \{C_1 \Delta C_2 : C_i \in C_0(M_i), (1 \leq i \leq 2)\}$. Then $C_0$ is the set of cycles of a binary matroid $M_1 \Delta M_2$ on $E_1 \cup E_2$.

(5.4.2) Three special cases of (5.1.7) are introduced by Seymour ([11] and [12]) as follows.

(i) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \Delta E_2|$, $M_1 \Delta M_2$ is a 1-sum of $M_1$ and $M_2$.

(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{z\}$, say, and $z$ is not a loop or coloop of $M_1$ or $M_2$, and $|E_1|, |E_2| < |E_1 \Delta E_2|$, $M_1 \Delta M_2$ is a 2-sum of $M_1$ and $M_2$.

(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, say, and $Z$ is a circuit of $M_1$ and $M_2$, and $Z$
includes no cocircuit of either $M_1$ or $M_2$, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a 3-sum of $M_1$ and $M_2$.

For $i = 1, 2, 3$, an $i$-sum of $M_1, M_2$ is denoted as $M_1 \oplus_i M_2$. The 1-sum $M_1 \oplus_1 M_2$ is also written as $M_1 \oplus M_2$.

(5.4.3) Let $M$ be a binary connected matroid. Suppose that $(X_1, X_2)$ (with $X_2 = E(M) - X_1$) is a 2-separation of $M$. Let $e$ be an element not in $E(M)$, $E_i = X_i \cup e$, ($1 \leq i \leq 2$). Define, for $i = 1, 2$,

$$C_i = \{ C \subseteq E_i : C \in \mathcal{C}(M) \text{ or for some } C' \in \mathcal{C}(M), C = C' \cap X_i \}.$$ 

Then each of the following holds.

(i) $C_i$ is the set of all circuits of a matroid $M_i$ on $E_i$.
(ii) $M = M_1 \oplus_2 M_2$.
(iii) $M_i$ is a proper minor of $M$.

(5.4.4) (Seymour [11]) Let $M$ be a binary connected matroid which does not have a 2-separation. Suppose that $(X_1, X_2)$ (with $X_2 = E(M) - X_1$) is a 3-separation of $M$ and such that $\min\{|X_1|, |X_2|\} \geq 4$. Then there exist proper minors $M_1$ and $M_2$ of $M$ such that $M = M_1 \oplus_3 M_2$.

(5.4.5) Let $R_{10}$ denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

(5.4.6) (Seymour’s Decomposition Theorem of Regular Matroids, [11]) Let $M$ be a regular matroid. If $M$ does not have 1, 2, or 3-separations, then either $M$ is graphic, or $M$ is cographic, or $M \cong R_{10}$.

(5.4.7) (Seymour [11]) Every regular matroid can be constructed by means of 1-, 2-, and 3-sums from graphic and cographic matroids, and copies of $R_{10}$. 

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6. Flows in Matroids

1. Flows in Networks

(6.1.1) Let $D$ be a digraph with a distinguished edge $e_0 = st$ such that $e$ is oriented from $t$ to $s$. Let $c : E(G) - \{e_0\} \to \mathbb{R}$ be a nonnegative function satisfying $c(e_0) = +\infty$. We call the triple $(D, e_0, c)$ a network and $c$ the capacity function of the network. For each $e \in V(G)$, $c(e)$ is the capacity of $e$.

(6.1.2) Let $A = (a_{ij}) = A(D)$ be the incidence matrix of the digraph $D$. That is,

$$a_{ij} = \begin{cases} 
1 & \text{if the edge } j \text{ is oriented from the vertex } i \\
-1 & \text{if the edge } j \text{ is oriented into the vertex } i \\
0 & \text{if the edge } j \text{ is not incident with the vertex } i
\end{cases}.
$$

The maximum flow problem of the network $(D, e_0, c)$ is to find a function $x : E \to \mathbb{R}$ such that (viewing $x$ and $c$ as vectors in the $|E|$-dimensional real vector space $V(|E|, \mathbb{R})$),

maximize $x(e_0)$,
subject to

$$Ax = 0 \text{ and for each } e \in E(G), 0 \leq x(e) \leq c(e).$$

The maximized value $x(e_0)$ is called the maximum flow value of the network $(D, e_0, c)$.

(6.1.3) The maximum flow problem can also be stated in terms of undirected graphs. Let $G$ be a graph with a distinguished edge $e = st$, where $s, t \in V(G)$. Let $c : E(G) - \{e_0\} \to \mathbb{R}$ be a nonnegative function satisfying $c(e_0) = +\infty$. We call the triple $(G, e_0, c)$ a undirected network with capacity $c$. Fix an orientation $D = (D(G))$ of $G$, and let $A = (a_{ij}) = A(D)$ be the incidence matrix of the digraph $D$.

The maximum flow problem of the undirected network $(G, e_0, c)$ is to find a function $x : E \to \mathbb{R}$ such that (viewing $x$ and $c$ as vectors in the $|E|$-dimensional real vector space $V(|E|, \mathbb{R})$),

maximize $x(e_0)$,
subject to

$$Ax = 0 \text{ and for each } e \in E(G), -c(e) \leq x(e) \leq c(e).$$
The maximized value \( x(e_0) \) is called the **maximum flow value** of the network \((D, e_0, c)\).

**Remark:** If we replace each edge \( e \) of \( G \) be a pair of oppositely oriented arcs \( e' \) and \( e'' \), incident with the same ends, and let the capacities of this pair of oppositely oriented arcs be the same as \( c(e) \), then Problem (6.1.3) becomes Problem (6.1.2). Problem (6.1.3) is usually called the **circular form** of the maximum flow problem of the undirected network \((G, e_0, c)\).

(6.1.4) Let \((G, e_0, c)\) be an undirected network. Define

\[
\mathcal{P} = \{ C - e_0 : e_0 \in C, \text{ and } C \in C(M(G)) \}.
\]

Members in \( \mathcal{P} \) are the \( e_0 \)-**paths** of \( G \). Denote \( E(G) = \{ e_0, e_1, \ldots, e_m \} \) and \( \mathcal{P} = \{ P_1, P_2, \ldots, P_n \} \). Define \( A = (a_{ij})_{n \times m} \) be the incidence matrix of elements in \( E(G) - \{ e_0 \} \) over \( \mathcal{P} \). That is, for \( 1 \leq j \leq m \) and \( 1 \leq i \leq n \),

\[
a_{ij} = \begin{cases} 
1 & \text{if } e_j \in P_i \\
0 & \text{if } e_j \notin P_i
\end{cases}.
\]

This matrix \( A \) is called the **incidence matrix** of the network \((G, e_0, c)\). Then, the **maximum flow problem** of the undirected network \((G, e_0, c)\) is to find a function \( y = (y_1, y_2, \ldots, y_n) \in V(n, \mathbb{R}) \) such that

\[
\text{Maximize } y \cdot 1 = y_1 + y_2 + \cdots + y_n \\
\text{subject to } yA \leq c \text{ and for each } 1 \leq i \leq n, y_i \geq 0.
\]

This is called the **path form** of the maximum flow problem of the undirected network \((G, e_0, c)\).

(6.1.5) (Ford and Fulkerson, [6]) For an undirected network \((G, e_0, c)\), if \( x \) is an optimal solution of Problem (6.1.3) and \( y \) is an optimal solution of Problem (6.1.4), then \( x(e_0) = y \cdot 1 = \sum_{i=1}^{n} y_i \).

(6.1.6) (Dual form of Problem (6.1.4)). Let \((G, e_0, c)\) be an undirected network. Define

\[
\mathcal{K} = \{ D - e_0 : e_0 \in D, \text{ and } D \in C(M^+(M)) \}.
\]
Members in $K$ are the $e_0$-cuts of $G$. Let $\mathcal{P}$ and $A$ be defined as in (6.1.4). Then each of the following holds.

(i) $\forall K \in \mathcal{K}$ and $P \in \mathcal{P}$, we have $K \cap P \neq \emptyset$.
(ii) If $y = (y_1, y_2, \cdots, y_n)$ satisfying $yA \leq c$, then

$$k = \min \left\{ \sum_{e \in K} c(e) : K \in \mathcal{K} \right\} \geq \sum_{i=1}^{n} y_i.$$ 

This value $k$ is called the minimum cut capacity of the undirected network $(G, e_0, c)$; and the set $K \in \mathcal{K}$ reaching the minimum capacity $k$ is a minimum cut of $(G, e_0, c)$.

(6.1.7) The Max-Flow-Min-Cut Theorem for undirected network: For undirected network $(G, e_0, c)$, each of the following holds:

(i) If a solution of Problem (6.1.4) is $y = (y_1, y_2, \cdots, y_n)$, then the maximum flow value $y \cdot 1 = \sum_{i=1}^{n} y_i$ (see Theorem (6.1.5)) equals the minimum cut capacity of $(G, e_0, c)$.

(ii) If $c$ is an integral capacity function, then Problem (6.1.4) also has an optimal integral solution.

(6.1.8) Let $M$ be a matroid with a distinguished element $e_0 \in E(M)$. Let $c : E(M) - \{e_0\} \rightarrow \mathbb{R}$ be a nonnegative function with $c(e_0) = \infty$. Then $(M, e_0, c)$ is called a matroid network. Define the set of the $e_0$-paths in $(M, e_0, c)$ by

$$\mathcal{P} = \mathcal{P}(M, e_0, c) = \{ C - e_0 : e_0 \in C, C \in \mathcal{C}(M) \},$$

and the set of the $e_0$-cuts in $(M, e_0, c)$ by

$$\mathcal{K} = \mathcal{K}(M, e_0, c) = \{ D - e_0 : e_0 \in D, D \in \mathcal{C}(M^*) \}.$$ 

The value $k = \min \left\{ \sum_{e \in K} c(e) : K \in \mathcal{K} \right\}$ is the minimum cut capacity of $(M, e_0, c)$, and the $e_0$-cut $K \in \mathcal{K}$ realizing this minimum is a minimum cut of $(M, e_0, c)$. We can define the incidence matrix of elements in $E(M) - \{e_0\}$ over $\mathcal{P}$ as in (6.1.4). (This matrix is called the incidence matrix of the matroid network $(M, e_0, c)$). Then the maximum flow problem in $(M, e_0, c)$ seeks a nonnegative real vector function $y = (y_1, y_2, \cdots, y_n)$ (where $n = |\mathcal{P}|$) such that

Maximize $y \cdot 1 = y_1 + y_2 + \cdots + y_n$

subject to

$yA \leq c$ and for each $1 \leq i \leq n, y_i \geq 0$. 

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For an optimal solution \( y \), the value \( \sum_{i=1}^{n} y_i \) is the **maximum flow value** of the matroid network \((M,e_0,c)\).

(6.1.9) Let \((M,e_0,c)\) be a matroid network. Let \( P = \mathcal{P}(M,e_0,c) \) and \( K = \mathcal{K}(M,e_0,c) \) be defined as in (6.1.8), and let let \( k \) be the minimum cut capacity of \((M,e_0,c)\). Let \( n = |\mathcal{P}| \). Each of the following holds.

(i) For each \( P \in \mathcal{P} \) and \( K \in \mathcal{K} \), \( P \cap K \neq \emptyset \).

(ii) If \( y \in V(n,\mathbb{R}) \) satisfying \( yA \leq c \), then \( k \geq y_1 + y_2 + \cdots + y_n \). In particular, the maximum flow value of \((M,e_0,c)\) does not exceed the minimum cut capacity of \((M,e_0,c)\).

**Proof:** (i) follows from Proposition (3.1.4). To prove (ii), we let \( y \in V(n,\mathbb{R}) \) satisfying \( yA \leq c \). Let \( K \in \mathcal{K}(M,e_0,c) \) be a minimum cut. (Thus \( k = \sum_{e \in K} c(e) \)). Let \( x_K \) be the characteristic function (viewed as a vector in \( V(|E|,\mathbb{R}) \)). By (i), \( Ax_K \geq 1 \). View \( c \) as a vector in \( V(|E|,\mathbb{R}) \). Then

\[
k = \sum_{e \in K} c(e) = c \cdot x_K \geq (yA) \cdot x_K = y \cdot (Ax_K) \geq y \cdot 1 = \sum_{i=1}^{n} y_i.
\]

2. Matroids with Max-Flow-Min-Cut Property

(6.2.1) Can Theorem (6.1.7) be extended to matroids? Here is an example. Let \( M = U_{2,4} \), 
\( E = E(M) = \{e_0,e_1,e_2,e_3\} \), and \( c = 1 \) be a constant function. Then

\[
\mathcal{P} = \mathcal{K} = \{\{e_1,e_2\},\{e_1,e_3\},\{e_2,e_3\}\}.
\]

Thus \( k = 2 \) and

\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

Add the three equations in \( yA \leq c \) side by side to get \( y_1 + y_2 + y_3 \leq \frac{3}{2} < 2 \). This means that the maximum flow value of \((M,e_0,c)\) cannot be the same as the minimum cut capacity of \((M,e_0,c)\).

(6.2.2) Let \( M \) be a matroid. If \( \forall e_0 \in E(M) \), for any non negative capacity function
$c : E(M) \to \mathbb{R}$, the maximum flow value is always equal to the minimum cut capacity in $(M, e_0, c)$, then we say that $M$ is a **matroid with maximum-flow-minimum-cut property** (abbreviated as a MFMC matroid). For an MFMC matroid $M$ with minimum capacity $k$, if whenever $c$ is integral, there is always an integral flow function $y = (y_1, y_2, \cdots, y_n)$ such that $\sum_{i=1}^{n} y_i = k$, then we say that $M$ is a **strong MFMC matroid**.

Theorem (6.1.7) indicates that all graphic matroids are strong MFMC matroids.

(6.2.3) Let $M$ be a matroid, and let $N$ be a minor of $M$. Each of the following holds.
(i) If $M$ is an MFMC matroid, then $N$ is also an MFMC matroid.
(ii) If $M$ is a strong MFMC matroid, then $N$ is also a strong MFMC matroid.

**Proof:** We shall prove that if $N^* = (M - e_0)^* = M^*/e_0$, if $K \in \mathcal{K}(M, e_0, c)$ is a minimum cut, then $(K \cup e_0) - f \in \mathcal{C}(N^*)$, and so

$$\sum_{e \in K - f} c(e) = \sum_{e \in K} c(e) = 1 \cdot y = 1 \cdot y_N.$$ 

This shows that $N = M - f$ is also an MFMC matroid.

(6.2.4) Note that $U_{2,4}$ is not binary. By Example (6.2.1), $U_{2,4}$ is not an MFMC matroid. By (6.2.3), an MFMC matroid cannot have $U_{2,4}$ as a minor. It follows by (5.3.6) that every MFMC matroid must be binary.
(6.2.5) Consider the binary matroid $F_7^* = M_2[A_1]$, where

$$
A_1 = \begin{bmatrix}
I_4 & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\end{bmatrix}.
$$

Let $e_0 \in E(F_7^*)$ be the label of the 7th column of $A_1$. Then we can compute the incidence matrix of $e_0$-paths in $F_7^*$, as follows:

$$
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}.
$$

We can also compute the incidence matrix of $e_0$-cuts in $F_7^*$, as follows:

$$
A^* = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
$$

Note that $F_7^*$ has 3 $e_0$-cuts of cardinality 2. Choose a capacity function $c = 1$ as a constant function. Then the capacity of minimum cuts is $k = 2$. Suppose that $y$ is a maximum flow. From $yA \leq c$ we have $3(1 \cdot y) = 6$, or $1 \cdot y = 2$. Thus in order for $1 \cdot y = k = 2$, we must have $yA = c$, and so $y_1, y_2, \ldots, y_6 = \frac{1}{2}$. It follows that $F_7^*$ is not a strong MFMC matroid.

(6.2.6) (Seymour [10]) A matroid $M$ is a strong MFMC matroid if and only if $M \in \text{EX}(U_{2,4}, F_7^*)$.

(6.2.7) (Gallai [7], Minty [8]) Every regular matroid is a strong MFMC matroid.

(6.2.8) Exercise: Consider the binary matroid $M = M_2[A_2]$, where

$$
A_2 = \begin{bmatrix}
I_4 & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{bmatrix}.
$$
Let $c = 1$ be the constant function. Show that for each $e_0 \in E(M)$, the maximum flow value is not the same as the capacity of a minimum cut in $(M, e_0, c)$.

References


