3.2 The Chain Rule

- We have seen one case of the chain rule already:
- Ex: If \( h(x) = (1-x^2)^{20} \), then
  \[
  h'(x) = 20(1-x^2)^{19}(-2x) = -40x(1-x^2)^{19}
  \]
The Generalized Power Rule

- If \( h(x) = [f(x)]^n \), the rule for this differentiation is

\[
    h'(x) = n[f(x)]^{n-1}f'(x)
\]

- This is sometimes called “The Generalized Power Rule”
Different “Outer Functions”

• Sometimes, the function $f(x)$ in $h(x) = [f(x)]^n$ has something done to it that’s not just a power
• Ex: $h(x) = 3(x-1)^2 + 2(x-1)^{10}$
• Ex: $h(x) = e^{3x-1}$ (whatever that is)
• Ex: $h(x) = \ln(x^2)$ (whatever that is)
The Chain Rule

• In general, if the “outer” function is $g(*)$ so that $h(x) = g(f(x))$, then

$$h'(x) = g'(f(x)) f'(x)$$

• This works even if $g(*)$ is not a power function
The Special Case

- The generalized power rule is a special case of the chain rule where \( g(f(x)) = (f(x))^n \)
- Ex: \( h(x) = (1-x^2)^3 \)

\[
\begin{align*}
  f(x) &= 1-x^2 \\
  g(x) &= x^3 \\
  h'(x) &= 3(1-x^2)^2(-2x) \\
  &= -6x(1-x^2)^2
\end{align*}
\]
Another Example

• Ex: Compute $h'(x)$ if $h(x) = g(f(x))$ and
  $f(x) = x^4 - x^2$  
  $g(x) = x^2 - 4$

\[
h(x) = g(f(x)) = g(x^4 - x^2) = (x^4 - x^2)^2 - 4
\]

$g'(x) = 2x$ and $f'(x) = 4x^3 - 2x$

\[
h'(x) = g'(f(x))f'(x)
\]

\[
= g'(x^4 - x^2)(4x^3 - 2x)
\]

\[
= (2(x^4 - x^2))(4x^3 - 2x)
\]
Related Rates of Change

• The notation $h'(x) = g'(f(x))f'(x)$ has another form. Sometimes it's convenient to let $u = f(x)$ so that $f'(x) = \frac{du}{dx}$ and then

$$h'(x) = \frac{dh}{dx} = \frac{dg}{du} \frac{du}{dx}$$
An Example

• What this means is that the rate of change of the whole function is the product of the rate of change of g with respect to u and the rate of change of u (=f(x)) with respect to x

• Ex: The cost of manufacturing $x$ cases of cereal is given by $C(x) = 3x + 4\sqrt{x} + 2$

Weekly production at $t$ weeks from present is estimated to be $x=6200 + 100t$ cases.
Example Continued

a) Find the marginal cost, \( \frac{dC}{dx} \).

\[
\frac{dC}{dx} = C'(x) = 3 + 4(\frac{d}{dx}(x^{1/2}))
\]

\[
= 3 + 4\left(\frac{1}{2}x^{-1/2}\right)
\]

\[
= 3 + \frac{2}{\sqrt{x}}
\]
Example Continued

b) Find the time rate of change of cost, \( \frac{dC}{dt} \).

\[
\frac{dC}{dt} = \frac{dC}{dx} \frac{dx}{dt} = C'(x(t))x'(t)
\]

\[
= (3 + 4\left( \frac{d}{dx}(x^{1/2}) \right))(100)
\]

\[
= (3 + 4\left( \frac{1}{2}(x(t))^{-1/2} \right))(100)
\]

\[
= (3 + \frac{2}{\sqrt{x(t)}})(100)
\]

\[
= (3 + \frac{2}{\sqrt{6200 + 100t}})(100)
\]