SUBGRAPHS WITH TRIANGULAR COMPONENTS*

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Corrádi and Hajnal [5] showed that if the minimum degree \( \delta(G) \) of a graph on \( p \) vertices satisfies the inequality \( \delta(G) \geq \frac{1}{3}(2p-1) \), then \( G \) has a subgraph consisting of \( \lceil \frac{1}{3}p \rceil \) triangular components. They gave a class of graphs which shows that their inequality is best possible. In this paper, we characterize the extremal graphs \( G \), and we thereby show that there are two classes of graphs \( G \) for which the inequality is best possible.

1. Introduction

Let \( G \) and \( H \) be graphs on \( p \) vertices such that

\[
(\Delta(\bar{G}) + 1)(\Delta(H) + 1) \leq p + 1,
\]

where \( \Delta \) denotes the maximum degree, and \( \bar{G} \) denotes the complement of \( G \). (We follow the notation of Harary [7].) We conjectured (see [2] or [3]) that (1.1) is sufficient to insure that \( H \) is a subgraph of \( G \), and we gave two classes of graphs to show that if the right side of (1.1) is replaced \( p + 2 \), then the conclusion fails for two classes of graphs. When \( \Delta(H) = 2 \), (1.1) is equivalent to

\[
\delta(G) \geq \frac{1}{3}(2p-1).
\]

Corrádi and Hajnal [5] gave the following result.

**Theorem 1.1.** Let \( G \) and \( H \) be graphs on \( p \) vertices, such that every component of \( H \) is a triangle, except possibly for one component that is either \( K_1 \) or \( K_2 \). If (1.2) holds, then \( H \) is a subgraph of \( G \).

The extremal graphs, i.e., those graphs \( G \) for which

\[
\delta(G) = \frac{1}{3}(p-1)
\]

and such that \( H \) is not a subgraph of \( G \) of Theorem 1.1 are the only known extremal graphs in the case \( \Delta(H) = 2 \) of the aforementioned conjecture.

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This result was announced in a paper presented at the 8th Southeastern Conference on Combinatorics.
It is often convenient to work with the complement \( \overline{G} \), rather than \( G \), because \( \overline{G} \) has fewer edges. Then \( H \) is a subgraph of \( G \) if and only if \( H \) and \( \overline{G} \) can be placed on the same vertex set with no overlapping edges.

The graphs in the first class of extremal graphs, referred to as type 1, are the graphs \( \overline{G} \) consisting of two cliques \( K_{b+1} \), where \( p = 3b + 1, \; b > 0 \). These cliques are components of \( \overline{G} \), since \( \delta(\overline{G}) = \frac{3}{2}(p-1) \) implies \( \Delta(\overline{G}) = \frac{1}{2}(p-1) = b \). We allow edges on the remaining \( p - 2(b + 1) = b - 1 \) vertices of a type 1 graph \( \overline{G} \). We also denote the class of type 1 graphs on \( p = 3b + 1 \) vertices by \( C_1(b) \).

The second class of extremal graphs (referred to generally as type 2 graphs), denoted \( C_2(b) \), consists of the graph on \( p = 3b + 1 \) vertices, with \( b \) odd, such that \( \overline{G} \) contains a clique \( K_{b+1} \) and a biclique \( K_{b,b} \). Since \( \delta(G) = \frac{3}{2}(p-1) \), these are separate components of \( \overline{G} \). Of course, \( |C_2(b)| = 1 \) for each odd \( b \).

It is readily verified that the graph \( H \), containing \( \lceil \frac{3}{4}p \rceil \) triangular components, is not a subgraph of a type 1 or a type 2 graph on \( p = 3b + 1 \) vertices.

We shall show that the type 1 and type 2 graphs are the only graphs with \( \delta(G) \geq \frac{3}{2}(p-1) \) which do not contain \( \lceil \frac{3}{4}p \rceil \) disjoint triangles.

2. Related literature

The following result of ours was announced in [1] (see footnote, page 226 of [1]) and it appears with proof in [2]. It was independently obtained by Sauer and Spencer [8].

**Theorem 2.1.** If two graphs \( G \) and \( H \), each on \( p \) vertices, satisfy

\[
2\Delta(H)\Delta(\overline{G}) \leq p - 1,
\]

then \( H \) is a subgraph of \( G \).

Theorem 2.1 agrees with conjecture if \( \Delta(H) \) or \( \Delta(\overline{G}) \) is 1, but otherwise the inequalities differ by a factor of almost 2.

Hajnal and Szemeredi [6] generalized Theorem 1.1, and they obtained the following result.

**Theorem 2.2.** If two graphs \( G \) and \( H \) on \( p \) vertices satisfy (1.1) and if \( H \) has \( \lfloor p/(\Delta(H)+1) \rfloor \) components isomorphic to \( K_{\Delta(H)+1} \), then \( H \) is a subgraph of \( G \).

The extremal graphs of Theorem 1.1 generalize to extremal graphs for Theorem 2.2: one merely uses components \( K_{b+1} \) and possibly a single \( K_{b,b} \) with \( b = \Delta(\overline{G}) \), instead.

Several other related results are mentioned in [3].

We shall need the following theorem (see [4]) to prove our main result. Let
$x_1 \in X_1$ and $x_2 \in X_2$ satisfy the requirement that
\[ f(X_1, X_2) = f(X_1 - x_1 + x_2, X_2 - x_2 + x_1), \]
then $x_1$ and $x_2$ are said to be interchangeable.

**Theorem 2.3.** Let $G$ be a graph with $p \geq 2$ and
\[ \delta(G) = c(p-1) \tag{2.2} \]
for some $c \in [0, 1)$. There is a nontrivial partition $X_1 \cup X_2$ of $V(G)$ which maximizes
\[ f(X_1, X_2) = \frac{1}{2}(1-c)(p_1^2 + p_2^2) - |E(G_1)| - |E(G_2)|, \tag{2.3} \]
where $G_1 = G[X_1]$ and $G_2 = G[X_2]$ are induced subgraphs, and $p_i = |X_i|$, for $i = 1, 2$. This partition satisfies
\[ \delta(G_i) \geq c(p_i - 1). \tag{2.4} \]
Furthermore, suppose $x_1 \in X_1$ and $x_2 \in X_2$ are adjacent in $G$ and satisfy
\[ \deg_{G_1}(x_1) + \deg_{G_2}(x_2) = \delta(G). \tag{2.5} \]
Then $x_1$ and $x_2$ are interchangeable,
\[ \deg_G(x_1) = \deg_G(x_2) = c(p-1), \tag{2.6} \]
and the vertices $X_{3-i}$ interchangeable with $x_i$ are adjacent in $G_{3-i}$ to $x_{3-i}$, for $i = 1$ and 2.

3. The main results

We shall prove the following two theorems:

**Theorem 3.1.** Let $G$ and $H$ be graphs on $p$ vertices, and suppose that every component of $H$ is isomorphic to either $K_1, K_2,$ or $K_3$. Let $b = b(H)$ denote the number of triangular components of $H$, and suppose $b \geq 0$. If
\[ \delta(G) \geq \lceil \frac{1}{2}(p+b) \rceil, \]
and if $H$ is not a subgraph of $G$, then either
\[ \text{There is a set } S \text{ of } b-1 \text{ vertices of } G \text{ such that } G-S \text{ is a complete bipartite graph; or} \tag{3.1} \]
\[ \text{There is a set } S \text{ of } b+1 \text{ vertices, } b \text{ odd, such that } G-S \text{ has two components, both isomorphic to } K_p, \text{ and } H \text{ has } \frac{1}{2}(p-1) \text{ triangles.} \tag{3.2} \]

**Theorem 3.2.** Let $G$ and $H$ be graphs on $p$ vertices and suppose that every component of $H$ is a triangle $K_3$, except for one vertex $K_1$ if $p = 3b + 1$, or one edge
If \( p = 3b + 2 \). If
\[
\delta(G) \geq \frac{3}{2}(p - 1),
\]
then \( H \) is not a subgraph of \( G \) if and only if both
\[
\delta(G) = \frac{3}{2}(p - 1) = 2b
\]
and \( G \) is of type 1 or type 2.

**Lemma 3.3.** Let \( G \) be a graph with \( p = 3b + 1 \) vertices, for some integer \( b \), and with \( \delta(G) \geq 2b \). If for some set \( S \subseteq V(G) \), with \( |S| = b - 1 \), \( G - S \) is bipartite, with bipartition \( V_1 \cup V_2 \), then the following conclusions hold.

Every vertex of \( S \) is adjacent to every vertex of \( G - S \):

\[
|V_1| = |V_2|;
\]

\( G - S \) is a complete bipartite graph.

Thus, \( G \) is of type 1.

**Proof.** Without loss of generality, assume that \( |V_1| \geq |V_2| \). We have
\[
|V_1| \geq \frac{1}{2}(p - |S|) = \frac{1}{2}(3b + 1 - (b - 1)) = b + 1.
\]

Let \( v_1 \in V_1 \). Since \( V_1 \cup V_2 \) is a bipartition of \( G - S \), \( v_1 \) is adjacent in \( \tilde{G} \) to every vertex of \( V_1 - v_1 \). But
\[
\Delta(\tilde{G}) = p - \delta(G) - 1 \leq b,
\]
and hence we must have \( |V_1| = b + 1 \) and \( \delta(G) = 2b \). Also, since \( \delta(G) \geq 2b \), each \( v_1 \in V_1 \) must be adjacent to every vertex of \( G - V_1 \), i.e., to every vertex of \( V_2 \) and every vertex of \( S \). The conclusions of the lemma follow directly.

**Remarks.** If \( G \) is of type 2, then \( p \equiv 4 \) (mod 6), and \( G \) is regular of degree \( 2b = \frac{3}{2}(p - 1) \). Note that the only graph that is both of type 1 and type 2 is the 4-cycle.

**Lemma 3.4.** Let \( G \) be a graph with \( p = 3b + 1 \) vertices, for some integer \( b \), and with \( \delta(G) \geq 2b \). If for some set \( S \subseteq V(G) \), with \( |S| = b + 1 \), \( G - S \) has two components, then the following conditions hold.

Every vertex of \( S \) is adjacent to every vertex of \( G - S \);
\( G - S \) has two components, both isomorphic to \( K_b \).
If, furthermore, \( b \) pairwise disjoint triangles do not embed in \( G \), then
\( p \equiv 4 \) (mod 6);
\( S \) is an independent set;
\( G \) is of type 1 only if \( G \) is a 4-cycle.
Thus, \( \tilde{G} \) is of type 2.

**Proof.** Let \( G \) and \( S \) satisfy the hypotheses. Since \( p = 3b + 1 \) and \( \delta(G) \geq 2b \), any
since $G - S$ has two components, any vertex in the smaller component is adjacent in $\tilde{G}$ to at least $\frac{1}{2} |V(G - S)| = b$ vertices in the larger component of $G - S$. But these statements force equality: both components have just $b$ vertices. Also, the first two conclusions of the lemma follow immediately.

If $S$ is not an independent set or if $p \not\equiv 4 \pmod{6}$, then either $G[S]$ has an edge, or, since $p = 3b + 1$, $p \equiv 1 \pmod{6}$. In either case, an embedding of $b$ pairwise disjoint triangles is easily found. The rest is easy.

**Proof of Theorem 3.1 from Theorem 3.2.** Assume without loss of generality that the components of $H$ consist of $b$ triangles $K_3$, $\frac{1}{2}(p - 3b)$ edges $K_2$, and $p - 3b - 2\frac{1}{2}(p - 3b)$ vertices $K_1$. By adding $\frac{1}{2}(p - 3b)$ vertices to $H$, each adjacent to both ends of a $K_2$, we can construct a graph $H'$ on $p + \frac{1}{2}(p - 3b)$ vertices, where the components of $H'$ consist of $b + \frac{1}{2}(p - 3b)$ triangles $K_3$ and $p - 3b - 2\frac{1}{2}(p - 3b)$ ($= 0$ or $1$) vertices $K_1$. By adding an independent set of $\frac{1}{2}(p - 3b)$ vertices to $G$, we construct a graph $G'$ in which each added vertex is adjacent to every vertex of $G$. Thus,

$$|V(G')| = p + \frac{1}{2}(p - 3b) = \frac{3}{2}(p - b),$$

and

$$\delta(G') \geq \min (p, \delta(G) + \frac{1}{2}(p - 3b))$$

$$\geq \min (p, \frac{1}{2}(p + b) + \frac{1}{2}(p - 3b))$$

$$= \min (p, 2\frac{1}{2}(p - b)) = 2\frac{1}{2}(p - b)$$

$$= \frac{3}{2}(3\frac{1}{2}(p - b)) \geq \frac{3}{2}(|V(G')| - 1).$$

Thus, by Theorem 3.2, either $H'$ is a subgraph of $G'$, or $G'$ is a graph of type 1 or type 2. Suppose $G'$ is a graph of type 2. If $G' \neq G$, then $G'$ has a vertex of degree $p$ and $|V(G')| < \frac{3}{2}p$. Hence, $G'$ is not a graph of type 2 unless $G' = G$. Then by Lemma 3.4,

$$|V(G')| \equiv 4 \pmod{6}.$$

In this case $\frac{1}{2}(p - 3b) = 0$ vertices were added to $G$ to get $G'$, whence $p - 3b = 1$, and $H$ has $b = \frac{1}{2}(p - 1)$ triangles, and we have the second case of Theorem 3.1.

Suppose $G'$ is a graph of type 1. Then $H'$ has

$$b' = b + \frac{1}{2}(p - 3b) = \frac{1}{2}(p - b)$$

triangles. Moreover, $|V(G')| = 3b' + 1$, and there is a set $S' \subseteq V(G')$, with $|S'| = b' - 1$, whose removal leaves a complete bipartite graph $G' - S' = K_{b' + 1, b' + 1}$. We have

$$\delta(G') \geq 2\frac{1}{2}(p - b) = 2b'.$$

We claim that $V(G') = V(G) \cup S'$. To prove this, suppose that $V(G) \cup S'$ does not contain a vertex $v \in V(G') - V(G)$. However, $V(G') - V(G)$ has only
vertices, and so some vertex $w$ of $G$ lies on the same side of the bipartition as $v$. But $v$ is adjacent to all vertices of $G$, and in particular to $w$, and we have a contradiction, which proves the claim.

Let $S = V(G) \cap S'$. Then by the claim,

$$|S| = |S'| - (|V(G')| - |V(G)|) = (b + \left\lceil \frac{1}{2}(p-3b) \right\rceil - 1) - \left\lceil \frac{1}{2}(p-3b) \right\rceil = b - 1,$$

and $G - S$ is bipartite. This is a conclusion of 3.1.

The remaining possibility is that $H'$ is a subgraph of $G'$. There is an embedding of $H'$ into $G'$ which extends an embedding of $H$ into $G$. This proves Theorem 3.1.

**Lemma 3.5.** Let $G$ be a graph, and $X_1 \cup X_2$ be partition of $V(G)$ of the type described in Theorem 2.3 for which

$$\delta(G_1) + \delta(G_2) = \delta(G). \quad (3.3)$$

Suppose that sets $Y_3 \subseteq X_1$, $V_3 \subseteq X_2$ exist such that

1. $G_2 - V_3$ contains a spanning complete bipartite subgraph with nontrivial bipartition $V_1 \cup V_2$; \quad (3.4)
2. $G_1 - Y_3$ is a complete bipartite graph with nontrivial bipartition $Y_1 \cup Y_2$; \quad (3.5)
3. If $v \in V_1 \cup V_2$ the $\deg_{G_2}(v) = \delta(G_2)$; \quad (3.6)
4. If $y \in Y_1 \cup Y_2$ then $\deg_{G_1}(y) = \delta(G_1)$. \quad (3.7)

Then any vertex of $Y_1 \cup Y_2$ is adjacent to every vertex in $V_j$, for some $j \in \{1, 2\}$. Suppose further that

No vertex of $Y_1 \cup Y_2$ is adjacent to vertices in both $V_1$ and $V_2$. \quad (3.8)

Then $G - (Y_3 \cup V_3)$ is a complete bipartite graph.

**Proof.** By (3.3), (3.6), and (3.7), the latter part of Theorem 2.3 may be applied to the vertices of $V_1 \cup V_2 \cup Y_1 \cup Y_2$.

Suppose that the first conclusion of the lemma is false for some $y \in Y_1 \cup Y_2$. Thus, $y$ is not adjacent in $G$ to a vertex $v_1$ of $V_1$ and a vertex $v_2$ of $V_2$. By Theorem 2.3, $v_1$ and $v_2$ are interchangeable with $y$, and are thus not adjacent in $G$. But, by (3.4), $v_1$ is adjacent to $v_2$. This contradiction proves the first part of the lemma.

By (3.8), any vertex $y \in Y_1 \cup Y_2$ is adjacent in $\bar{G}$ to every vertex $V_j$ for some $j \in \{1, 2\}$. By the first part of the lemma, which was just proved, $y$ is adjacent in $G$ to every vertex of $V_{3-j}$.

Thus, the vertices of $Y_1 \cup Y_2$ fall into two classes: those, the set of which we denote $V$, which are adjacent (in $G$) to vertices of $V_1$ but not $V_2$; and those, the
Let \( Y_4 \subset Y_4 \). Since \( y_4 \) is interchangeable with, but not adjacent to, each vertex of \( V_2 \), Theorem 2.3 shows that \( V_2 \) is actually an independent set. Similarly, so is \( V_1 \). Therefore \( V_1 \cup V_2 \) induces a complete bipartite graph in \( G_2 \).

We claim that \( \{ Y_4, Y_5 \} = \{ Y_1, Y_2 \} \). To see this, suppose that \( Y_4 \cap Y_1 \) and \( Y_4 \cap Y_2 \) are both nonempty. Then any vertex \( v_2 \in V_2 \) is not adjacent in \( G \) to a vertex \( y_1 \in Y_4 \cap Y_1 \), nor to a vertex \( y_2 \in Y_4 \cap Y_2 \). By Theorem 2.3, \( y_1 \) and \( y_2 \) are interchangeable with \( v_2 \) and are thus not adjacent. However, (3.5) implies that \( y_1 \) and \( y_2 \) are adjacent. This contradiction shows that either \( Y_4 \cap Y_1 \) or \( Y_4 \cap Y_2 \) is empty. Similarly, either \( Y_5 \cap Y_1 \) or \( Y_5 \cap Y_2 \) is empty. Since \( V_1 \cup V_2 \) and \( Y_1 \cup Y_2 \) are nontrivial, and

\[
Y_4 \cup Y_5 = Y_1 \cup Y_2,
\]

the claim must follow.

In either case of this claim, there is a \( j \in \{1, 2\} \) such that \( (V_1 \cup Y_j) \cup (V_2 \cup Y_{3-j}) \) is a bipartition of \( G - (Y_3 \cup V_3) \), and this bipartite graph is complete. This proves Lemma 3.5.

We define for \( X_j' \subset V(G) \)

\[
G_j' = G[X_j'] \quad j = 1, 2,
\]

and

\[
p_j' = |X_j'|, \quad j = 1, 2.
\]

A vertex \( v \) of \( G, G_j \) or \( G_j' \) is critical in \( G, G_j, G_j' \), if

\[
\deg_G(x) - 1 < \frac{1}{2}(p-1),
\]

\[
\deg_{G_j}(x) - 1 < \frac{1}{2}(p_j - 1),
\]

or

\[
\deg_{G_j'}(x) - 1 < \frac{1}{2}(p_j' - 1),
\]

respectively.

**Lemma 3.6.** Suppose Theorem 3.2 is valid for all graphs with fewer than \( p \) vertices. Suppose

\[
p \equiv 1 \pmod{3}
\]

and that \( X_1 \cup X_2 \) is a partition of \( V(G) \) which satisfies the conditions of Theorem 2.3 with \( c = \frac{3}{8} \). For \( \{z, z'\} \subset V(G) \), write

\[
X_j' = X_j - \{z, z'\} \quad \text{for } j = 1, 2,
\]

and assume that

\[
p_j' \equiv 1 \pmod{3} \quad \text{for } j = 1, 2,
\]

that

\[
\delta(G_j') \geq \frac{2}{3}(p_j' - 1) \quad \text{for } j = 1, 2,
\]

(3.9)
and that $p_j \equiv 0 \pmod{3}$ for $j \in \{1, 2\}$ implies that $z \in X_j$ and that there exist critical vertices $x_3, x_4 \in X_{j-1}$ such that $G[z, x_3, x_4]$ is a triangle. Then, if $\frac{1}{3}(p_j' - 1)$ pairwise disjoint triangles cannot be embedded in $G_j'$, for $j = 1$ and $j = 2$, both $G_1'$ and $G_2'$ are of type 1.

**Proof.** Since Theorem 3.2 holds for graphs on fewer than $p$ vertices, since (3.9) holds, and since $\frac{1}{3}(p_j' - 1)$ triangles cannot be embedded in $G_j'$, $j = 1, 2$, it follows that $G_1'$ is of type 1 or type 2, and $G_2'$ is of type 1 or type 2. Thus

$$\delta(G_j') = \frac{2}{3}(p_j' - 1), \quad j = 1, 2,$$

whence

$$\delta(G_1') + \delta(G_2') = \frac{2}{3}(p_1' + p_2' - 2) = \frac{2}{3}(p - 1) - 2. \quad (3.10)$$

Moreover, by Theorem 2.3,

$$\delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p - 1) - \frac{2}{3}.$$ The left side is an integer and $p \equiv 1 \pmod{3}$, whence

$$\delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p - 1).$$

So that (3.10) also holds, it follows that if $z$ or $z'$, respectively, is in $X_j$, for $j \in \{1, 2\}$, then $z$ or $z'$ is adjacent in $G$ to every critical vertex of $G_j'$. In fact

$$\delta(G_1) + \delta(G_2) = \frac{2}{3}(p - 1), \quad (3.11)$$

and vertices critical in $G_j'$ are critical in $G$, for $j = 1, 2$. Also, since critical vertices of $G_1$ and critical vertices of $G_2$ are interchangeable if they are adjacent in $G$, critical vertices of $G_1'$ and critical vertices of $G_2'$ are also interchangeable if they are adjacent in $G$. By (2.6) of Theorem 2.3, such vertices are also critical in $G$.

Without loss of generality, it suffices to show that $G_2'$ is of type 1.

Let $y_1, y_2$ be any pair of adjacent critical vertices of $G_1'$. If $G_1'$ is of type 2, then every vertex of $G_1'$ is critical in $G_1'$, whence, any adjacent pair suffices. If $G_1'$ is of type 1, then $p_1' \equiv 4$, and there is a set $Y_3 \subseteq X_1'$ with

$$|Y_3| = \frac{1}{3}(p_1' - 1) - 1,$$

such that $G_1' - Y_3$ is a complete bipartite graph with bipartition $Y_1 \cup Y_2$, where

$$|Y_1| = |Y_2| = \frac{1}{3}(p_1' - 1) + 1.$$ Since $Y_1 \cup Y_2$ is the set of critical vertices in $G_1'$ if $y_1 \in Y_1$ and $y_2 \in Y_2$, then $y_1$ and $y_2$ are adjacent critical vertices of $G_1'$.

Suppose by way of contradiction that $G_2'$ is of type 2 and not of type 1. Then every vertex $v$ of $G_2'$ is critical in $G_2$, and hence interchangeable with $y_i (i = 1, 2)$ if $y_i$ is adjacent in $G$ to $v$. 


Since \( y_i \) is critical in \( G'_1 \) and in \( G \),
\[
|E(y_i, X'_2)| = \deg_G(y_i) - \deg_{G'_1}(y_i) - |E(y_i, \{z, z'\})| = \frac{2}{3}(p - 1) - \frac{2}{3}(p'_1 - 1) - |E(y_i, \{z, z'\})| = \frac{2}{3}(p'_2 + 2) - |E(y_i, \{z, z'\})|.
\]
Hence, the number of vertices of \( G'_2 \) adjacent in \( \tilde{G} \) to \( y_i \) is at least
\[
p'_2 - |E(y_i, X'_2)| = \frac{1}{3}(p'_2 - 1) + |E(y_i, \{z, z'\})| - 1,
\]
and these vertices are interchangeable with \( y_i \) and thus form an independent set.

We have two cases: when \( \{z, z'\} \cap X_1 \) is not empty, and when \( \{z, z'\} \subseteq X_2 \). In the first case, without loss of generality, suppose \( z \in X_1 \). In \( G_1 \), \( z \) is adjacent to every critical vertex of \( G'_1 \), including \( y_1, y_2 \in X'_1 \). Hence, \( |E(y_i, \{z, z'\})| \geq 1 \). In the second case, by the hypotheses of the lemma, \( p_2 \equiv 0 \) (mod 3) and \( z \) lies in a triangle \( G[z, x_3, x_4] \), where \( x_3 \) and \( x_4 \) are adjacent critical vertices of \( G_1 \). Pick \( y_1, y_2 \) so that \( \{y_1, y_2\} = \{x_3, x_4\} \), which is possible because \( G_1 = G'_1 \) here. Then \( |E(y, \{z, z'\})| \geq 1 \). Therefore, in either case there are at least \( \frac{1}{3}(p'_2 - 1) \) critical vertices of \( G'_2 \) interchangeable with \( y_i \) (\( i = 1, 2 \)).

Since \( G'_2 \) is of type 2, \( p'_2 \equiv 4 \) (mod 6), and since \( G'_2 \) is not of both type 1 and type 2, it follows that \( p'_2 \geq 10 \). Hence, at least \( \frac{1}{3}(p'_2 - 1) \geq 3 \) critical vertices of \( G'_2 \) are interchangeable with \( y_i \) (\( i = 1, 2 \)). By Theorem 2.3, this set of \( \frac{1}{3}(p'_2 - 1) \) vertices is an independent set. But since \( G'_2 \) is of type 2, there is only one maximal independent set \( S_2 \) of more than 2 vertices, and \( S_2 \) has \( \frac{1}{3}(p'_2 - 1) + 1 \) vertices. Therefore, each \( y_i \) (\( i = 1, 2 \)) is interchangeable with all but at most one vertex of \( S_2 \). Since \( |S_2| \geq 3 \), there is a critical vertex \( v \in S_2 \), critical in \( G_2 \), interchangeable with both vertices \( y_1, y_2 \) critical in \( G_1 \). By Theorem 2.3 \( y_1 \) and \( y_2 \) are not adjacent, contrary to the choice of \( y_1 \) and \( y_2 \). Hence, \( G'_2 \) is of type 1, and the lemma is proved.

We leave to the reader the proofs of the next two lemmas.

**Lemma 3.7.** Let \( G_0 \) be a graph of type 1 on \( 3b_0 + 1 \) vertices. Let \( S_0 \) be the set of \( b_0 - 1 \) vertices whose removal leaves \( G - S_0 = K_{b_0 + 1, b_0 + 1} \). Any embedding of \( b_0 - 1 \) pairwise disjoint triangles into \( G_0 \) uses all but four vertices \( v_1, v_2, v_3, v_4 \in V(G_0) - S_0 \), and these four vertices induce a 4-cycle in \( G_0 \). Furthermore, \( v_1, v_2, v_3, v_4 \) may be chosen to be any four vertices of \( G_0 - S_0 \) that induce a 4-cycle in \( G \).

**Lemma 3.8.** Let \( G_0 \) be a graph of type 2 on \( 3b_0 + 1 \) vertices. Let \( S_0 \) be the independent set of \( b_0 + 1 \) vertices such that \( G_0 - S_0 \) consists of two components, each \( K_{b_0} \). Any embedding of \( b_0 - 1 \) pairwise disjoint triangles into \( G_0 \) uses all but four vertices, two in \( S_0 \), and one in each \( K_{b_0} \), and these four vertices induce a 4-cycle in \( G_0 \). Furthermore, for any four vertices of \( V(G_0) \) with two in \( S_0 \) and one in each \( K_{b_0} \), there is an embedding of \( b_0 - 1 \) pairwise disjoint triangles into the remaining \( 3b_0 - 3 \) vertices of \( G_0 \).
To save work, we assume without proof Theorem 1.1 of Corrádi and Hajnal [5]:

**Proof of Theorem 3.2.** By Theorem 1.1, it suffices to consider graphs $G$ for which

$$\delta(G) = \frac{2}{3}(p-1).$$

Equality implies that

$$p \equiv 1 \pmod{3}.$$ 

Thus, we can assume that $H$ is a graph with $b$ triangles and one isolated vertex, and that

$$p = 3b + 1, \quad \delta(G) = \frac{2}{3}(p-1) = 2b.$$ 

By Theorem 2.3 there are disjoint nonempty sets $X_1, X_2$ such that $V(G) = X_1 \cup X_2$ and the induced subgraphs $G_i$, for $G_i = G[X_i]$, $i = 1, 2$, satisfy

$$\delta(G_i) \geq \frac{2}{3}(p_i - 1), \quad (3.12)$$

where $p_i = |X_i|$.

Assume inductively that Theorem 3.2 is true for graphs smaller than $G$, and suppose that $H$ is not a subgraph of $G$. Theorem 3.2 is true for $p \leq 4$, and so we have a basis for induction. We have two cases: either one of the sets $X_i$ has cardinality a multiple of 3, or neither do. In one subcase (Subcase IIA), we show that if $H$ is not a subgraph of $G$, then $G$ is of type 2. In other subcases, we verify the hypotheses of Lemma 3.5, and hence there is a subset $S = Y_3 \cup V_3$ of $V(G)$, with $|S| = b - 1$, such that $G - S$ is a bipartite graph. Thus, by Lemma 3.3, $G$ is of type 1. We consider each case below.

**Case I.** Suppose that

$$p_1 \equiv 0 \pmod{3} \quad \text{and} \quad p_2 \equiv 1 \pmod{3}.$$ 

Since $\delta(G_1)$ is an integer and $p_1 \equiv 0 \pmod{3}$, (3.12) gives

$$\delta(G_1) \geq \frac{2}{3}(p_1 - 1) + \frac{2}{3} = \frac{2}{3}p_1,$$

and Theorem 1.1 implies that $\frac{1}{3}p_1$ triangles can be embedded in $G_1$. Write $b_1 = \frac{1}{3}p_1$ and

$$b_2 = \frac{1}{3}(p_2 - 1), \quad (3.13)$$

and note that

$$b_1 + b_2 = b,$$

and that the $b_1 \geq 1$ triangles embedded in $G_1$ use each vertex of $G_1$. Since $b$ triangles are assumed not to embed in $G$, it follows that $b_2$ triangles do not embed in $G_2$. By the induction hypothesis, either $G_2$ is of type 1, and there is a set
$V_3 \subseteq X_2$ with

$$|V_3| = b_2 - 1$$

(3.14)

such that

$$G_2 - V_3 = K_{b_2+1, b_2+1},$$

or $G_2$ is of type 2 and there is an independent set

$$S_2 \subseteq X_2$$

(3.15)

on $b_2+1$ vertices, such that $G_2 - S_2$ has two components, each a clique on $b_2$ vertices.

If $G_2$ is of type 2, each vertex $v \in X_2$ has degree

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1).$$

If this alternative applies, write

$$V_2 = S_2, \quad V_1 = G_2 - S_2,$$

and note that (3.4) and (3.6) of Lemma 3.5 hold with $V_3$ empty.

If $G_2$ is of type 1, let

$$V_1 \cup V_2$$

denote the bipartition of $G_2 - V_3$.

Then (3.4) of Lemma 3.5 holds,

$$|V_1| = |V_2| = b_2 + 1,$$

and so (3.12) and (3.13) give

$$\delta(G_2) \geq 2b_2,$$

which allows us to apply Lemma 3.3. Also, by Lemma 3.3, each vertex of $V_i (i = 1, 2)$ is adjacent to every vertex of $V_{3-i}$ and to every vertex of $V_3$, and if $v \in V_1 \cup V_2$,

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1)$$

whence (3.6) holds if $G_2$ is of type 1.

It follows in either alternative (either $G_2$ of type 1 or type 2) that there must be at least

$$\deg_G(v) - \deg_{G_2}(v) \geq \frac{2}{3}(p - 1) - \frac{2}{3}(p_2 - 1) = \frac{2}{3}p_1$$

vertices in $X_1$ adjacent to a given vertex $v \in V_1 \cup V_2$, and that (3.4) and (3.6) hold.

Denote by $N(v_1, v_2)$ the vertices of $X_1$ that are adjacent to both $v_1 \in V_1$ and $v_2 \in V_2$. We have

$$|N(v_1, v_2)| \geq 2\left(\frac{2}{3}p_1\right) - p_1 = b_1.$$

(3.18)

Since $G_2$ is of type 1 or type 2, $b_2$ disjoint triangles do not embed in $G_2$. By Lemmas 3.7 and 3.8, there is an embedding of $b_2 - 1$ pairwise disjoint triangles
into $G_2$ such that the four remaining vertices induce a 4-cycle in $G_2$, with two of its vertices in $V_1$ and the other two in $V_2$. Let $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ be disjoint edges of this 4-cycle, where $v_1, v'_1 \in V_1$ and $v_2, v'_2 \in V_2$.

In the two subcases below, we establish that the hypotheses of Lemma 3.5 apply to $G_1$ and $G_2$. We have already established (3.4) and (3.6), and it remains to establish (3.3), (3.5), (3.7), and (3.8).

**Subcase IA.** Suppose that distinct vertices $v_1, v'_1 \in V_1$ and $v_2, v'_2 \in V_2$ exist such that $\{N(v_1, v_2), N(v'_1, v'_2)\}$ possesses a transversal $\{y, y'\}$ in $X_1$, i.e., distinct $y, y' \in X_1$ such that

$$y \in N(v_1, v_2), \quad y' \in N(v'_1, v'_2)$$

Since

$$\delta(G_1) \geq \frac{2}{3}p_1,$$

we have

$$\delta(G_1 - \{y, y'\}) \geq \frac{2}{3}p_1 - 2 = \frac{2}{3}(p_1 - |\{y, y'\}| - 1).$$

Since $b$ pairwise disjoint triangles do not embed in $G$, and since $(b_2 - 1) + 2$ triangles can be embedded in $G[X_2 \cup \{y, y'\}]$, we cannot embed

$$b - (b_2 + 1) = b_1 - 1$$

triangles in $G_1 - \{y, y'\}$. By the induction hypotheses, $G_1 - \{y, y'\}$ is a graph of type 1 or of type 2, and by Lemma 3.6 with $\{y, y'\} = \{z, z'\}$, and with $\{v_1, v_2\} = \{x_3, x_4\}$, both $G_1 - \{y, y'\}$ and $G_2$ are of type 1. Therefore, there is a set $Y_3'$ of $b_1 - 2$ vertices such that $G_1 - Y_3$ is bipartite, where

$$Y_3 = Y_3' \cup \{y, y'\}.$$  \hspace{1cm} (3.19)

Let $Y_1 \cup Y_2$ be the bipartition of $G_1 - Y_3$. By definition,

$$|Y_1| = |Y_2| = b_1 = \frac{1}{3}p_1,$$

and by Lemma 3.3, any vertex $y_j$ of $Y_j (j = 1, 2)$ is adjacent to every vertex of $Y_{3-j} \cup Y_3$ and has degree $\frac{2}{3}p_1 - 2$ in $G_1 - \{y, y'\}$. Thus, (3.5) holds. Since

$$\deg_{G_1}(y_j) \geq \delta(G_1) = \frac{2}{3}p_1,$$

each vertex of $Y_i$ is also adjacent to $y$ and $y'$, and hence has degree $\delta(G_1)$ in $G_1$, whence we have (3.7). Therefore

$$\delta(G_1) + \delta(G_2) = \frac{2}{3}p_1 + \frac{2}{3}(p_2 - 1) = \frac{2}{3}(p - 1) = \delta(G),$$

which is (3.3). We have thus proved (3.3) through (3.7) of Lemma 3.5.

We prove (3.8) by assuming that it is false: thus, there is a vertex $y'' \in Y_1 \cup Y_2$ adjacent to some $v'' \in V_1$ and to some $v''_2 \in V_2$. By the first conclusion of Lemma 3.5, $y''$ is adjacent to all vertices in either $V_1$ or in $V_2$. Without loss of generality, suppose that $y''$ is adjacent to all vertices in $V_2$. 
By reconsidering \( \{N(v_1, v_2), N(v'_1, v'_2)\} \) of this subcase, we shall produce a pair of such sets which have either \( \{y, y''\} \) or \( \{y', y''\} \) as a transversal. If \( v''_1 = v_1 \), then choose \( v''_1 = v_2 \in V_2 \) as another vertex to which \( y'' \) is adjacent. Then \( \{y, y''\} \subseteq N(v_1, v_2) \), and \( y'' \in N(v_1, v_2), y' \in N(v'_1, v'_2) \) will be a transversal. On the other hand, if \( v''_1 \neq v_1 \), then choose \( v'_2 \neq v_2, v'_2 \in V_2 \). Then \( y \in N(v_1, v_2), y'' \in N(v'_1, v'_2) \) will be a transversal.

There is no loss of generality in using \( y', y'' \) for our transversal in the following argument. We shall produce a vertex \( y^* \) whose degree in \( G_1 \) is less than \( \delta(G_1) = \frac{2}{3}p_1 \), and this will be our desired contradiction.

Using the transversal \( \{y', y''\} \), we repeat the argument of this subcase. Thus, there is a set \( Y'' \), instead of \( Y_3 \), such that \( |Y''_3| = b_1 \), \( G_1 - Y''_3 \) is bipartite, and \( y', y'' \in Y'' \). Since \( y'' \notin Y_3 \) and since \( |Y_3| = |Y_3''| \), there is a vertex \( y^* \in Y_3 - Y_3'' \). Note that \( G_1 - \{y', y''\} \) is of type 1, just like \( G_1 - \{y, y'\} \).

By previous remarks of this subcase,

\[
\deg_{G_1}(y_j) = \delta(G_1) = \frac{2}{3}p_1,
\]

for every \( y_j \in Y_j, j = 1 \) or 2. Since \( y'' \in Y_1 \cup Y_2 \), \( \deg_{G_1} y'' = \frac{2}{3}p_1 \), and when we repeat the argument of this subcase using the transversal \( \{y', y''\} \), the vertex \( y^* \in Y_3 - Y''_3 \) has the same property:

\[
\deg_{G_1} y^* = \frac{2}{3}p_1.
\]

However, \( y^* \), being in \( Y_3 \), is adjacent to all \( \frac{2}{3}p_1 \) members of \( Y_1 \cup Y_2 \), and hence is not adjacent to \( y' \in Y_3 \). Thus, \( y^* \in V(G_1 - Y''_3) \) is not adjacent to all \( b_1 = \frac{1}{3}p_1 \) members of its own side of the bipartition of \( G_1 - Y_3 \), and is not adjacent to \( y' \in Y'' \), whence \( \deg_{G_1} y^* < \frac{2}{3}p_1 \), a contradiction. Thus, (3.8) must hold.

**Subcase IB.** Suppose that there is no pair of disjoint vertices \( v_1, v'_1 \in V_1, v_2, v'_2 \in V_2 \) in \( G_2[V_1 \cup V_2] \) such that \( \{N(v_1, v_2), N(v'_1, v'_2)\} \) possesses a transversal.

Since \( p_1 > 0 \), (3.18) implies that \( b_1 \geq 1 \) and that \( N(v_1, v_2) \) and \( N(v'_1, v'_2) \) are nonempty. Since \( \{N(v_1, v_2), N(v'_1, v'_2)\} \) possesses no transversal, we have \( y \in X_1 \) such that

\[
N(v_1, v_2) = y = N(v'_1, v'_2).
\]

Thus, by (3.18) \( b_1 = 1 \) and \( p_1 = 3 \). Since \( \delta(G_1) = \frac{2}{3}p_1 \), we see that \( G_1 \) is a triangle, and

\[
\delta(G_1) = 2 = \frac{2}{3}p_1,
\]

which is condition (3.7) of Lemma 3.5. Define

\[
Y_3 = \{y\}, \tag{3.20}
\]

and let each of the other two vertices of \( G_1 \) be put in separate singleton sets \( Y_1, Y_2 \). Then (3.5) of Lemma 3.5 is valid. We have

\[
\delta(G_1) + \delta(G_2) = 2 + \frac{2}{3}(p_2 - 1) = \frac{2}{3}p_1 + \frac{2}{3}(p_2 - 1) = \frac{2}{3}(p - 1) = \delta(G),
\]

which proves (3.3).
Finally, we must show that (3.8) of Lemma 3.5 holds in this subcase. Suppose that (3.8) is false, and that there is a vertex $y'' \in Y_1 \cup Y_2$ such that $G[v''_1, v''_2, y'']$ is a triangle in $G$, where $v''_1 \in V_1$ and $v''_2 \in V_2$. Then $\{v''_1, v''_2\}$ overlaps $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$, for otherwise, we would be in Subcase IA, because $\{N(v''_1, v''_2), N(v_1, v_2)\}$ or $\{N(v'_1, v'_2), N(v'_1, v'_2)\}$ would have a transversal. Without loss of generality, suppose that $v''_1 = v_1$ and $v''_2 = v_2$. Then $N(v'_1, v_2) = y''$, because otherwise $\{N(v_1, v'_2), N(v'_1, v_2)\}$ would have a transversal, and we would be in Subcase IA. But then $v_1, v_2, v'_1, v'_2$ are not only adjacent to $y \in Y_3$ of Subcase IB; they are also adjacent to $y'' \in Y_1 \cup Y_2$, and so $\{N(v_1, v_2), N(v'_1, v'_2)\}$ possesses the transversal $\{y, y''\}$, and IA applies. Thus, (3.8) must hold if we are to avoid this contradiction. This completes Subcase IB.

Thus, all of the hypotheses of Lemma 3.5 hold. We conclude from Lemma 3.5 that $G - (Y_3 \cup V_3)$ is a bipartite graph. Thus, $\hat{G} - (Y_3 \cup V_3)$ consists of two cliques. By (3.14), (3.19), and (3.20), we have

$$|Y_3 \cup V_3| = b - 1,$$

and so by Lemma 3.3, $G$ is of type 1. This completes Case I.

**Case II.** Suppose that

$$p_1 \equiv p_2 \equiv 2 \pmod{3}.$$ 

Since $\delta(G_i)$ is an integer, (3.12) implies

$$\delta(G_i) \geq \frac{2}{3}p_i - \frac{1}{3}$$

(3.22)

for $i = 1, 2$. Without loss of generality, assume

$$p_1 \leq p_2.$$ 

Write

$$b_1 = \frac{1}{3}p_1 - \frac{3}{3}, \quad b_2 = \frac{1}{3}p_2 - \frac{3}{3},$$

and note that $b_1$ and $b_2$ are integers such that

$$b_1 + b_2 + 1 = b.$$ 

If we form a graph $G_i + z$, adding to $G_i(i = 1, 2)$ a new vertex $z$ adjacent to every vertex of $G_i$, then by (3.22),

$$\delta(G_i + z) = \frac{2}{3}|X_i + z|,$$

and by Theorem 1.1, $b_i + 1$ pairwise disjoint triangles can be embedded in $G_i + z$. Therefore, $b_i$ pairwise disjoint triangles and an edge disjoint from the $b_i$ triangles, which we shall call the free edge, can be embedded in $G_i$, for $i = 1, 2$. We shall attempt to use the vertices of the two free edges to form an extra triangle, disjoint from the $b_i$ triangles in $G_1$ and the $b_2$ triangles in $G_2$, thus constituting $b_1 + b_2 + 1 = b$ pairwise disjoint triangles in $G$. By assuming that $b$ pairwise disjoint
triangles do not embed in $G$, we shall determine the structure of $G$ in the attempt to find such an embedding.

We show in the two subcases below that either $G$ is of type 2, or there is a vertex $x_3 \in X_2$ such that the free edge in $G_1$ together with $x_3$ form a triangle in $G$. It may be necessary to alter the embedding of $b_1$ triangles and the free edge into $G_1$ in order to accomplish this.

Let $x_1, x_2$ be the ends of the free edge in $G_1$. Without loss of generality, choose the free edge from among all possible free edges so that

$$\text{deg}_{G_1}(x_1) + \text{deg}_{G_1}(x_2)$$

is minimized. If $x_1$ and $x_2$ are adjacent in $G$ to a vertex $x_3 \in X_2$, then $x_1, x_2, x_3$ is the desired triangle. Otherwise, $x_1$ and $x_2$ are adjacent to no common vertex in $X_2$. Then

$$\text{deg}_{G_1}(x_1) + \text{deg}_{G_1}(x_2) \geq 2\delta(G) - p_2$$

$$\geq \frac{2}{3}(p - 1) - p_2 = p_1 + \frac{1}{3}p - \frac{4}{3}.$$  \hspace{1cm} (3.23)

Also, without loss of generality, assume that

$$\text{deg}_{G_1}(x_1) \geq \text{deg}_{G_1}(x_2).$$

These inequalities imply

$$2 \text{deg}_{G_1}(x_1) \geq \text{deg}_{G_1}(x_1) + \text{deg}_{G_1}(x_2) \geq p_1 + \frac{1}{3}p - \frac{4}{3}.$$  \hspace{1cm} (3.24)

We define

$$\pi : V(H_1) \rightarrow V(G_1)$$

to be an embedding of $b_1$ triangles $K_3$ and one edge-component $K_2$, constituting $H_1$, into $G_1$ such that the edge-component $K_2$ is mapped to the free edge $x_1, x_2$ that minimizes $\text{deg}_{G_1}(x_1) + \text{deg}_{G_1}(x_2)$. We shall alter $\pi$ if necessary, and then either we shall extend $\pi$ to an embedding of $H$ into $G$, where $H$ consists of $b$ triangular components and one isolated vertex, or we shall show (Subcase IIA) that $G$ is of type 2 or (following the subcases) that $G$ is of type 1.

Define

$$M(x) = \{x' \in X_1 : \pi^{-1}(x) \text{ and } \pi^{-1}(x') \text{ are adjacent in } H_1\}.$$  For $i = 1, 2$, and $x \in V(G)$, define

$$N_i(x) = \{x' \in X_i : x \text{ and } x' \text{ are adjacent in } G\}.$$

We say that $x \in X_1$ is a successor of $x_1 \in X_1$ if each vertex of $M(x_1)$ is adjacent in $G_1$ to $x$. Denote the set of successors of $x_1$ by $S(x_1)$. We say that $x_1 \in X_1$ is a predecessor of $x \in X_1$ if $x$ is a successor of $x_1$. Denote the set of predecessors of $x$ by $P(x)$.

Subcase IIA. We adopt the following notation: $(x_1x_4)'$ denotes the transposition $(x_1x_4)$ of $x_1$ and $x_4$ in $X_1$ if $x \neq x_4$; $(x_1x_4)'$ denotes the identity permutation if $x_1 = x_4$. 
Suppose that
\[ \deg_{G_1}(x_2) \leq \frac{1}{3}(p-1). \]

First, we eliminate the possibility of strict inequality. If the inequality above is strict, then
\[ |E(x_2, X_2)| = \deg_G(x_2) - \deg_{G_1}(x_2) > \frac{2}{3}(p-1) - \frac{1}{3}(p-1) = \frac{1}{3}(p-1). \]

Since \( x_1 \) is not adjacent to at most \( \frac{1}{3}(p-1) \) vertices of \( G \) other than \( x_1 \), it is adjacent to one of the more than \( \frac{1}{3}(p-1) \) vertices \( x_3 \) of \( X_2 \) incident with an edge of \( E(x_2, X_2) \). Hence \( G[x_1, x_2, x_3] \) is a triangle on the free edge in \( G_1 \) and a vertex of \( G_2 \).

Henceforth in this subcase, we shall suppose
\[ \deg_{G_1}(x_2) = \frac{1}{3}(p-1). \]

By (3.23),
\[ \deg_{G_1}(x_1) + \frac{1}{3}(p-1) \geq p_1 + \frac{1}{3}p - \frac{4}{3}. \]

Hence,
\[ \deg_{G_1}(x_1) \geq p_1 - 1, \]
and so \( x_1 \) must be adjacent to each vertex of \( G_1 \). Therefore, \( P(x_1) = G_1 - x_2 \). Since \( S(x_1) = N_1(x_2) \), we conclude that for any \( x_4 \in N_1(x_2) \), \( (x_1x_4)\pi \) is an embedding of the \( b_1 \) triangles and free edge into \( G_1 \). Note that the embedding \( (x_1x_4)\pi \) makes \( \{x_4, x_2\} \) the free edge. By the minimality of \( \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \),
\[ \deg_{G_1}(x_4) + \deg_{G_1}(x_2) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2), \]
whence,
\[ \deg_{G_1}(x_4) = p_1 - 1. \]

Since \( x_4 \) may be any of the \( \frac{1}{3}(p-1) \) vertices of \( N_1(x_2) \), we know that the vertices of \( X_1 - N_1(x_2) \) must be adjacent to each vertex of \( N_1(x_2) \), a set of \( \frac{1}{3}(p-1) \) vertices adjacent to all of \( G_1 \). Hence,
\[ \delta(G_1) \geq \frac{1}{3}(p-1) = \deg_{G_1}(x_2). \]

Define the sets
\[ T_1 = N_1(x_2), \quad T_2 = N_2(x_2), \]
\[ S_1 = X_1 - T_1, \quad S_2 = X_2 - T_2. \]

We have already shown that \( G[T_1] \) is a complete graph, and each vertex of \( S_1 \) is adjacent to every vertex of \( T_1 \). If there is an \( x_4 \in T_1 = S(x_1) \) and a vertex \( x_3 \in X_2 \) such that \( G[x_2, x_3, x_4] \) is a triangle in \( G \), then we have accomplished the goal of this subcase, since \( (x_1x_4)\pi \) is an embedding of \( b_1 \) triangles and a disjoint edge.
mapped to \( \{x_2, x_4\} \), which is the edge forming the triangle with \( x_3 \). Otherwise, no \( x_4 \in T_1 \) forms a triangle with \( x_2 \) and any vertex in \( X_2 \). Hence, no \( x_4 \in T_1 \) is adjacent to vertices of \( T_2 \). Now,

\[ |T_2| = \deg_{G_1}(x_2) - \deg_{G_1}(x_3) \geq \frac{1}{3}(p - 1), \]

and hence, any \( x_4 \in T_1 \), having degree at least \( \frac{2}{3}(p - 1) \) in \( G \), must be adjacent to every vertex of \( S_1 \cup T_1 \cup S_2 - x_4 \). A similar argument shows that any vertex of \( T_2 \), not being adjacent to any vertex of \( T_1 \), a set of \( \frac{1}{3}(p - 1) \) vertices, is adjacent to any vertex of \( S_1 \cup T_2 \cup S_2 \) except itself. Note that this implies that \( G[T_2] \) is like \( G[T_1] \), a complete graph on \( \frac{1}{3}(p - 1) \) vertices. Also, note that any vertex of \( S_1 \cup S_2 \) is adjacent to every vertex of \( T_1 \cup T_2 \) in \( G \).

Hence, \( S_1 \cup S_2 \) is a set of \( |V(G) - (T_1 \cup T_2)| = p - \frac{2}{3}(p - 1) = \frac{1}{3}(p - 1) + 1 = b + 1 \) vertices whose removal from \( G \) leaves two components \( G[T_i], i = 1, 2 \), each a complete graph on \( \frac{1}{3}(p - 1) = b \) vertices.

By Lemma 3.4, either \( b \) pairwise disjoint triangles embed in \( G \), or \( G \) is of type 2. The first possibility is contrary to hypothesis. The other possibility is a desired conclusion of Theorem 3.1. Hence, we can assume that there is a free edge in \( G_1 \), which together with some \( x_3 \in X_2 \), forms a triangle in \( G \).

Subcase IIB. Suppose that

\[ \deg_{G_1}(x_2) \geq \frac{1}{3}(p - 1). \]  \hspace{1cm} (3.25)

Let \( x_3 \) be a vertex of \( X_2 \) that is adjacent in \( G \) to \( x_2 \). Since \( p_1 \leq p_2 \),

\[ \deg_{G_1}(x_2) \geq \frac{2}{3}(p - 1) \geq \frac{2}{3}(2p_1 - 1) = p_1 + \frac{2}{3}p_1 - \frac{2}{3} > p_1 - 1, \]

and so \( x_3 \) exists. The successors \( S(x_1) \) of \( x_1 \) in \( G_1 \) are the vertices of \( G_1 \) adjacent to \( x_2 \). We see that \( S_1(x_1) = N_1(x_2) \). We have

\[ |S(x_1) \cap N_1(x_3)| \geq \deg_{G_1}(x_2) + \deg_{G_1}(x_3) - (p_2 - 1) - |S(x_1) \cup N_1(x_3)| \]
\[ \geq \deg_{G_1}(x_2) + \frac{2}{3}(p - 1) - (p_2 - 1) - p_1 \]
\[ = \deg_{G_1}(x_2) - \frac{1}{3}(p - 1) > 0, \]

by (3.25). Hence, there is a vertex \( x_4 \in X_1 \) that forms a triangle with \( x_2 \) and \( x_3 \) and is a successor of \( x_1 \).

If \( x_1 \in S(x_4) \), then the embedding \( (x_1, x_4) \pi \) maps the free edge in \( G_1 \) to \( \{x_2, x_4\} \), which forms with \( x_3 \in X_2 \) a triangle in \( G \) as desired. Otherwise,

\[ x_1 \notin S(x_4). \]  \hspace{1cm} (3.26)

We shall find a vertex \( x_5 \in X_1 \) with \( x_5 \in S(x_4) \cap P(x_1) \), whence \( (x_1, x_4, x_5) \pi \) is the desired embedding of \( b_1 \) triangles and one edge into \( G_1 \).

In the image of the triangle embedded into \( G_1 \) having vertex \( x_4 \) are two other vertices, which we call \( x_6, x_7 \). The successors of \( x_4 \) are those vertices in \( G_1 \).
adjacent to both \(x_6\) and \(x_7\). Hence, \(x_1, x_6, x_7 \notin S(x_4)\), and
\[
|S(x_4)| \geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1. \tag{3.27}
\]

The predecessors \(P(x_i)\) of \(x_i\) in \(G_1\) are those vertices \(v \in X_1\) such that \(x_i\) is adjacent to all vertices of \(M(v)\). Now, \(x_1\) is adjacent in \(\bar{G}_1\) to \(p_1 - \deg_{G_1}(x_1) - 1\) vertices \(v' \in X_1\). Any such \(v'\) lies in exactly two sets \(M(v), v \in X_1\). Thus \(x_1 \notin S(v)\) for at most
\[
2p_1 - 2 \deg_{G_1}(x_1) - 2
\]
vertices \(v\) of \(X_1 - M(x_1) = G_1 - x_2\). Since the remaining vertices of \(G_1 - x_2\) are in \(P(x_1)\), we have \(x_2 \notin P(x_1)\), and
\[
|P(x_1)| \geq |X_1 - x_2| - (2p_1 - 2 \deg_{G_1}(x_1) - 2)
\]
\[
= 2 \deg_{G_1}(x_1) - p_1 + 1 \geq \frac{1}{3}(p - 1), \tag{3.28}
\]
by (3.24).

Suppose first that \(x_4\) is not adjacent to \(x_1\). Then
\(x_2, x_6, x_7 \notin P(x_1)\),
and we combine (3.27), (3.28), (3.22), and \(2p_1 \leq p\) to get
\[
x_1, x_2, x_6, x_7 \notin S(x_4) \cap P(x_1), \quad x_6, x_7 \notin S(x_4) \cup P(x_1),
\]
and
\[
|S(x_4) \cap P(x_1)| \geq |S(x_4)| + |P(x_1)| - |X_1 - \{x_6, x_7\}|
\]
\[
\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 + \frac{1}{3}p - \frac{1}{3} - p_1 + 2
\]
\[
\geq 2\delta(G_1) - 2p_1 + \frac{1}{3}p + \frac{2}{3}
\]
\[
\geq 2\left(\frac{1}{3}p - \frac{1}{3}\right) - 2p_1 + \frac{1}{3}p + \frac{2}{3}
\]
\[
= \frac{1}{3}p - \frac{2}{3}p_1 + 1 \geq 1. \tag{3.29}
\]

Suppose, otherwise, that \(x_4\) is adjacent to \(x_1\). Then \(G[x_1, x_2, x_4]\) is a triangle, and \(\{x_6, x_7\}\) is a free edge. Thus, by choice of \(\{x_1, x_2\}\) and (3.23),
\[
\deg_{G_1}(x_6) + \deg_{G_1}(x_7) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2)
\]
\[
\geq p_1 + \frac{1}{3}p - \frac{2}{3}. \tag{3.30}
\]
We combine (3.27), (3.28), (3.30) and
\[p_1 + p_2 = p\]
to obtain
\[
|S(x_4) \cap P(x_1)| \geq |S(x_4)| + |P(x_1)| - p_1
\]
\[
\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 + \frac{1}{3}p - \frac{1}{3} - p_1
\]
\[
\geq p_1 + \frac{1}{3}p - \frac{4}{3} - 2p_1 + \frac{1}{3}p - \frac{1}{3}
\]
\[
= \frac{2}{3}p - \frac{3}{3}p_1 - \frac{1}{3}p_1 - \frac{5}{3}
\]
\[
\geq \frac{2}{3}p_2 - \frac{1}{3}p_1 - \frac{5}{3}
\]
\[
= \frac{1}{3}(p_2 - p_1) + \left(\frac{1}{3}p_2 - \frac{5}{3}\right). \tag{3.31}
\]
Note that both of the terms in the last line of (3.31) are nonnegative if \( p_2 \geq 5 \), and if \( p_2 > 5 \), then the last line is positive. If \( p_2 \leq 5 \), then \( p_1 \leq p_2 \) and \( p_1 \equiv 2 \pmod{3} \) imply one of the following three cases:

\[
p_2 = p_1 = 5;
\]

\[
p_2 = 5, \quad p_1 = 2;
\]

or

\[
p_2 = p_1 = 2.
\]

If \( p_2 = p_1 = 5 \), then (3.23) gives

\[
\deg_{G_1}(x_1) + \deg_{G_1}(x_2) \geq 7,
\]

whence \( \deg_{G_1}(x_1) \geq \deg_{G_1}(x_2) \) implies that \( x_1 \) is adjacent to every vertex of \( G_1 \) except itself, whence \( x_4 \in P(x_1) \), in violation of (3.26). If \( p_2 = 5, p_1 = 2 \), then the last line of (3.31) is 1, which is as desired. If \( p_1 = p_2 = 2 \), then \( p = 4 \) and \( \delta(G) \geq \frac{2}{3}(p - 1) \) imply \( G \) is \( K_4, K_4 - e \) (an edge), or a quadrilateral, all of which satisfy the theorem. Hence, under our hypotheses, the last line of (3.31) and the last line of (3.29) may be assumed to be positive.

Therefore, whether or not \( x_4 \) and \( x_1 \) are adjacent, there is a vertex \( x_5 \neq x_1 \) or \( x_2 \), such that

\[
x_5 \in S(x_4) \cap P(x_1),
\]

and so we have a closed alternating chain in \( G_1 \) represented by the permutation

\[
\alpha = (x_1 x_4 x_5).
\]

Hence, \( \alpha \pi \) is an embedding of the \( b_1 \) triangles and one edge into \( G_1 \). The free edge is determined by \( \alpha \pi \) to be \( \{x_2, x_4\} \), since \( x_1 \) is permuted to \( x_4 \) and since \( x_2 \neq x_5 \) guarantees that \( x_2 \) is fixed. Thus, the free edge is part of a triangle \( G[x_2, x_3, x_4] \), as desired. \textit{This concludes Subcase II.B.}

To complete Case II and the proof of the theorem, we verify that all the hypotheses, and hence the final conclusion, of Lemma 3.5 apply to \( G_1 \) and \( G_2 \), and then we show that \( G \) is of type 1.

Since we have assumed that

\[
b = b_1 + 1 + b_2
\]

triangles do not embed in \( G \), and since \( b_1 + 1 \) triangles embed in \( G_1 + x_3 = G[X_1 + x_3] \), we know that we cannot embed \( b_2 \) triangles in \( G_2 - x_3 \). Now,

\[
|V(G_2) - x_3| = 3b_2 + 1,
\]

and by (3.22),

\[
\delta(G_2 - x_3) \geq \delta(G_2) - 1 \geq \frac{2}{3}p_2 - \frac{1}{3} - 1
\]

\[
= \frac{2}{3}(|X_2 - x_3| - 1) = 2b_2.
\]
Therefore, by the induction hypothesis, \( G_2 - x_3 \) is of type 1 or type 2. Hence,

\[
\delta(G_2 - x_3) = \frac{2}{3}(p_2 - 2) = \frac{2}{3}p_2 - \frac{4}{3},
\]

whence, by (3.22), \( x_3 \) is adjacent to every vertex of \( G_2 - x_3 \) having degree

\[ \frac{2}{3}p_2 - \frac{4}{3} = \delta(G_2 - x_3) \]

in \( G_2 - x_3 \).

By Lemmas 3.7 and 3.8, we know that \( b_2 - 1 \) triangles embed in \( G_2 - x_3 \), and that such an embedding uses all but 4 vertices of \( G_2 - x_3 \). Moreover, these 4 vertices all have degree \( \delta(G_2 - x_3) \) in \( G_2 - x_3 \), and they induce a 4-cycle. Now, \( x_3 \) is adjacent to all four of these vertices, and hence forms a triangle with 2 of them.

Let \( v_1 \) and \( v_2 \) denote the other 2 vertices on this 4-cycle. Note that \( v_1 \) and \( v_2 \) are adjacent. We shall show that there are \( b_1 \) choices of a vertex \( y_3 \in X_1 \) such that \( G[v_1, v_2, y_3] \) is a triangle. If \( b_1 \) disjoint triangles can be embedded in \( G_1 - y_3 \), then, counting the triangle containing \( x_3 \), the triangle \( G[v_1, v_2, y_3] \), and the \( b_2 - 1 \) triangles of \( G_2 - x_3 \), we have \( b \) pairwise disjoint triangles in \( G \), contrary to assumption. Hence, \( b_1 \) triangles do not embed in \( G_1 - y_3 \). For this to happen,

\( b_1 \geq 1 \).

Thus, by (3.22), we may apply Lemma 3.6, with

\[
\{z, z'\} = \{x_3, y_3\},
\]

and conclude that both \( G_1 - y_3 \) and \( G_2 - x_3 \) are of type 1. Since

\[
\delta(G) = 2b = 2b_1 + 2 + 2b_2,
\]

and since

\[
deg_{G_2}(v_j) = 2b_2 + 1 \quad (j = 1, 2),
\]

each \( v_j \) \( (j = 1, 2) \) is adjacent to at least \( 2b_1 + 1 \) vertices of \( G_1 \). Hence, there are at least

\[
|E(v_1, X_1)| + |E(v_2, X_1)| - p_1 \geq 2(2b_1 + 1) - (3b_1 + 2) = b_1 \geq 1
\]

choices \( y_3 \in X_1 \) such that \( G[v_1, v_2, y_3] \) is a triangle. Therefore, as we already remarked, we may apply Lemma 3.6 and conclude that both \( G_1 - y_3 \) and \( G_2 - x_3 \) are of type 1.

Next, we establish the hypotheses of Lemma 3.5.

Since \( G_1 - y_3 \) and \( G_2 - v_3 \) are of type 1, where

\[
v_3 = x_3,
\]

and since they have \( 3b_1 + 1 \), \( 3b_2 + 1 \) vertices, respectively, there are sets \( Y'_3 \subseteq X_1 - y_3 \) and \( V'_3 \subseteq X_2 - v_3 \) with

\[
|Y'_3| = b_1 - 1, \quad |V'_3| = b_2 - 1,
\]

such that \( G_1 - y_3 - Y'_3 \) and \( G_2 - v_3 - V'_3 \) are complete bipartite graphs \( Y_1 \cup Y_2 \) and \( V_1 \cup V_2 \), respectively. Define

\[
Y_3 = Y'_3 + y_3, \quad V_3 = V'_3 + v_3.
\]
Thus, (3.4) and (3.5) of Lemma 3.5 hold, and also
\[ |Y_3 \cup V_3| = b_1 + b_2 = b - 1. \] (3.32)
Since \( G_1 - y_3 \) is of type 1, if \( y \in Y_1 \cup Y_2 \), then
\[ \deg_{G_1 - y_3}(y) = 2b_1 = \frac{2}{3}(p_1 - 2). \]
Now \( \delta(G_1) \geq \frac{2}{3}p_1 - \frac{1}{3} \), and hence \( y \) is adjacent to \( y_3 \in X_1 \). Therefore, for any \( y \in Y_1 \cup Y_2 \),
\[ \deg_{G_1}(y) = \frac{2}{3}(p_1 - 2) + 1 = \delta(G_1), \]
and (3.7) of Lemma 3.5 is established. Similarly, since \( G_2 - v_3 \) is of type 1, (3.6) may be established, and also for any \( v \in V_1 \cup V_2 \),
\[ \deg_{G_2}(v) = \delta(G_2). \]
By (3.22),
\[ \delta(G_1) + \delta(G_2) \geq \frac{2}{3}p_1 - \frac{1}{3} + \frac{2}{3}p_2 - \frac{1}{3} = \frac{2}{3}(p - 1) = \delta(G), \]
and (3.3) is established. Thus, having proved (3.3) through (3.7) of Lemma 3.5, we conclude from Lemma 3.5 that any vertex \( y \in Y_1 \cup Y_2 \) is adjacent to every vertex in \( V_j \) for some \( j \in \{1, 2\} \).
Suppose by way of contradiction that some \( y \in Y_1 \cup Y_2 \) is adjacent in \( G \) to vertices \( v_1 \in V_1 \) and \( v_2 \in V_2 \) (i.e., suppose that (3.8) is false). Thus, \( G[y, v_1, v_2] \) is a triangle. By Lemma 3.7, for any vertices \( v_3 \in V_1 - v_1 \) and \( v_4 \in V_2 - v_2 \), there is an embedding of \( b_2 - 1 \) triangles into \( G_2 - \{v_1, v_2, v_3, v_4, v_5\} \), since \( G_2 - v_3 \) is of type 1. Note that \( G[v_3, v_4, v_5] \) is also a triangle. We conclude from Lemma 3.7 that for any vertices \( y_1 \in Y_1, y_2 \in Y_2 \), there is an embedding of \( b_1 - 1 \) pairwise disjoint triangles into \( G_1 - \{y, y_1, y_2, y_3\} \), since \( G_1 - y_3 \) is of type 1. Including the \( b_2 - 1 \) triangles of \( G_2 - \{v_1, v_2, v_3, v_4, v_5\} \) and the 3 triangles \( G[y_1, y_2, y_3], G[y, v_1, v_2], \) and \( G[v_3, v_4, v_5] \), we have
\[ (b_1 - 1) + (b_2 - 1) + 3 = b \]
airwise disjoint triangles embedded in \( G \), contrary to assumption. Hence, (3.8) holds, and by Lemma 3.5, \( G - (Y_3 \cup V_3) \) is a complete bipartite graph. By (3.32) and Lemma 3.3, \( G \) is of type 1. This completes the proof of Theorem 3.2.

References

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